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Energy change due to the appearance of cavities in elastic solids

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Abstract

The paper presents an overview of the problem of assessing an increment of strain energy due to the appearance of small cavities in elastic solids. The following approaches are discussed: the compound asymptotic method by Mazja et al., the Eshelby-like method used in the classical works on the mechanics of composites, the homogenization method, and the topological derivative method proposed by Sokołowski and Żochowski. The increment of energy is expressed by a quadratic form with respect to strains referring to the virgin solid. All the methods lead to the same formula for the increment of energy. It is expressed by a quadratic form with respect to strains referring to the virgin solid. This quadratic form turns out to be unconditionally positive definite. Explicit formulae are derived for an elliptical hole and for a spherical cavity. The results derived determine the characteristic function of the *bubble* method of the optimal shape design of elastic 2D and 3D structures.

Keywords: Cavity problem; Bubble method; Topological derivative

1. Introduction

The problem of assessing the change in energy stored in an elastic body weakened by a small cavity is of fundamental importance in the mechanics of porous media. Although its averaged solution can be found by the energy method of Eshelby (1957), originally developed for the ellipsoidal inclusion case, it traces back to older results of Mackenzie (1950), concerning inclusions and cavities of spherical shapes. Knowing a formula for change of energy due to the appearance of a cavity or an inclusion makes it possible to assess effective moduli of composites. The simplest solutions concern the case of non-interactive inhomogeneities. These solutions based on Eshelby's results are well known to the community dealing with the mechanics of composites (cf. Christensen, 1979; Mura, 1982; Nemat-Nasser and Hori, 1993). It seems, however, that the asymptotic justification of Eshelby's methods, developed in Mazja et al. (1991) and Maz'ya and Nazarov

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(1987), is less known or even ignored. A deeper insight into the small opening problem reveals a link between Eshelby's formulae and older solutions of Polya and Szegö, see Schiffer and Szegö (1949). These results interrelate the small opening problem with the problem of a concentrated loading and, consequently, with the theory of Green's function. On the other hand, the results of Polya and Szegö opened new perspectives for posing and solving some optimization questions. The works of Mazja et al. extended the Polya–Schiffer–Szegö results to the elasticity field thus providing asymptotic justifications of Eshelby's formulae. Also these results contribute to formulating new classes of shape optimization problems (see Nazarov and Sokołowski, 2003).

The classical shape design problem reads: lay out a given amount of material in a given feasible domain such that the overall compliance of the structure attains a minimum. It is known that this formulation requires relaxation. If based upon the homogenization theory, such relaxation admits a new design material of porous microstructure, the properties of which being determined by the homogenization formulae (cf. Allaire and Kohn, 1993; Cherkaev, 2000; Lewiński and Telega, 2000, and the literature cited therein). In the case of a dilute distribution of cavities within a representative volume element, the effective properties are governed by simplified formulae found previously by Eshelby (1957) without any reference to mathematical averaging methods. These formulae can be further linearized with respect to porosity density without any loss in accuracy. On the other hand, the same results can be justified by the asymptotic methods of Mazja et al. (1991). Therefore, the following four approaches complement each other: Eshelby (mechanics of composites), Mazja et al. (asymptotic analysis of perturbation of domains), Sokołowski and Żochowski (topological derivative of shape functionals) and the homogenization approach.

Relaxation by homogenization is not the only method to attack shape optimization problems. Important suboptimal solutions have been found by the *bubble* method of Eschenauer et al. (1994) and Schumacher (1995) cf. Eschenauer and Schumacher (1997). The idea of its numerical algorithm consists in removing these subdomains where a characteristic function assumes values smaller than a given threshold value. It has turned out recently that the characteristic function of the bubble method is determined by the *topological derivative* of shape functionals. Sokołowski and Żochowski (1999a,b) presented an algorithm for finding the topological derivative of a large class of shape functionals, whereas the only shape functional considered in Eschenauer et al. (1994) was the compliance. Let us recall that the topological derivative defined in Sokołowski and Żochowski (1999a,b) represents a change of a given shape functional caused by the appearance of a small spherical opening in a given domain. Lewiński and Sokołowski (1999, 2000) generalized the notion of the topological derivative to the case of non-spherical openings in the context of the Neumann boundary value problem, and for the energy functional. This generalization made use of the compound asymptotic expansions of Mazja et al. (1991). Further generalizations to a more general class of functionals can be found in Nazarov and Sokołowski (2003). New numerical techniques of the bubble method have been recently developed by Garreau et al. (2001).

In the present paper the notion of a directional topological derivative is defined for appearing of non-circular holes and non-spherical cavities in general case of linear elasticity. The functional of compliance is considered. A proof is given that the characteristic function of the bubble method for the compliance functional coincides with the expression for a change of the elastic energy caused by the appearance of an arbitrary hole in the 2D setting, and by the appearance of cavity of an arbitrary shape in the 3D setting. In this problem the results of the following methods coincide: the Eshelby like approaches (see Kachanov et al., 1994) the compound asymptotics method and the topological derivative method.

The paper is organized as follows. In Section 2 we rederive the results of Mazja et al. (1991) concerning the change of energy caused by the appearance of a small hole in an elastic solid. We derive the formulae for the Polya–Szegö tensor $\boldsymbol{M}$, treated here as the rank four tensor. Our approach differs essentially from that of Mazja et al. (1991). The formula for the energy change is then rearranged to a form applicable in the mechanics of composites (see Kachanov, 1999). In particular, the cavity compliance tensor $\boldsymbol{H}$ is introduced in a new manner. In Section 4 we refer to the concept of the topological derivative for circular holes and prove that this method leads to the tensor $\boldsymbol{H}$ identical with that followed from the Eshelby-like approach. Sensitivity of the energy functional with respect to the appearance of a non-circular hole is considered in Section 5 by generalization of the topological derivative approach. With using the Muskhelishvili (1975) solution it is shown that all available methods result in the same formula for the energy change due to the appearance of an elliptical hole. The topological derivative determines the sensitivity of energy by a quadratic form with a matrix $\boldsymbol{G}$. It has turned out recently that $\boldsymbol{G} = \boldsymbol{M}$ irrespective of the shape of a hole (see Nazarov and Sokołowski, 2003). We note that the matrix $\boldsymbol{M}$ is unconditionally negative definite. This proof is straightforward (see Section 2.3). On the contrary, the very definition of the matrix $\boldsymbol{G}$ suggests that it is negative definite for star-shaped domains only (see Eq. (161)). Due to the equality $\boldsymbol{G} = \boldsymbol{M}$ we realize that this assumption is redundant. The explicit formulae for the components of $\boldsymbol{G} = \boldsymbol{M}$ can be found for an elliptical hole, as shown in Section 6. Independent computations of components of $\boldsymbol{M}$ and $\boldsymbol{G}$ are reported for the convenience of the reader.

In Section 7 we put forward a generalization of previous results to the 3D setting. We refer to the Eshelby-like results for a spherical cavity. Then we show the way the cavity compliance tensor $\boldsymbol{H}$ appears in the compound asymptotics approach. According to Nazarov and Sokołowski (2003) this result is equivalent to the result found by the topological derivative method.

The effective moduli of composites of an isotropic matrix and anisotropic inclusions of small concentration have been derived by Sanchez-Palencia (1985) in the context of a scalar elliptic problem with periodic coefficients. Similar approximate formulae for the effective moduli of porous media with periodically distributed voids can be put forward within the linear elasticity formulation as exposed e.g. by Jikov et al. (1994). In Section 8 we prove that in the case of dilute distribution of spherical cavities the homogenization formulae simplify to the formulae which are linearization of the Eshelby equations with respect to the density of porosity. Then we show that these formulae coincide with those reported in Nemat-Nasser and Hori (1993), where they follow from the method of prescribing microstrains. On the other hand, the method in which microstresses are prescribed can be viewed as a result of imposing the dilute approximation on the homogenization formulae put in their dual form, involving homogenized stresses.

Applications of the dilute approximation formulae for effective moduli extend the framework of the mechanics of composites. Since relaxation by homogenization means admitting a porous body as a design material, the dilute approximation formulae apply in the algorithm of shape optimization, where the porosity density plays the role of a new design variable.

The problems considered in the present paper should not be misled with the cavitation problem referring to finite deformations of elastic solids, discussed in the papers by Ball (1982), Müller and Spector (1995) and in the papers cited therein.

The following notation will be adopted. Small Latin indices, like $i, j, k, l, m, n, p, q, s, \ldots$ run over 1, 2, 3; the small Greek indices, like α, β, λ, μ, γ, δ, ι, κ, …, except for ε, assume the values 1, 2. The symbol ε represents a small positive parameter. Let $\boldsymbol{u} = (u_1, u_1)$ and $\boldsymbol{w} = (w_1, w_2, w_2)$ depend on (x_1, x_2), (y_1, y_2) or (x_1, x_2, x_3), (y_1, y_2, y_3), respectively. Then we define the operators

$$\epsilon_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right), \quad \epsilon^y_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial y_\beta} + \frac{\partial u_\beta}{\partial y_\alpha}\right),$$

$$\epsilon_{ij}(\boldsymbol{w}) = \frac{1}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right), \quad \epsilon^y_{ij}(\boldsymbol{w}) = \frac{1}{2}\left(\frac{\partial w_i}{\partial y_j} + \frac{\partial w_j}{\partial y_i}\right).$$

Since all tensors are referred to Cartesian systems the position of indices is arbitrary. The summation convention applies to the indices at different levels. To avoid misunderstandings the sign $\sum$ appears, if necessary.

The space of symmetric tensors of the rank two is denoted by $\mathbb{M}^2_s$, while $\mathbb{M}^4_s$ is the space of all tensors of the rank four possessing well known symmetry properties of the Hooke's tensor. The inner product in $\mathbb{M}^2_s$ is defined by ":"; thus for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{M}^2_s$ we write $\boldsymbol{a} : \boldsymbol{b} = a^{\alpha\beta} b_{\alpha\beta}$ or $a^{ij} b_{ij}$. The norm is defined by $\|\boldsymbol{a}\| = (\boldsymbol{a} : \boldsymbol{a})^{1/2}$.

For $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^4_s$ the product $\boldsymbol{AB}$ is defined by $(\boldsymbol{AB})^{\alpha\beta\lambda\mu} = (A^{\alpha\beta\gamma\delta} B^{\lambda\mu}_{\gamma\delta})$ in the 2D case and, similarly, for the 3D case.

The unit tensors $\mathbb{1} \in \mathbb{M}^2_s$ and $\mathbb{I} \in \mathbb{M}^4_s$, referred to the 3D Euclidean space spanned over the orthonormal basis $(\boldsymbol{e}_i)$ have the following representations

$$\mathbb{1} = \delta^{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j, \tag{1}$$

$$\mathbb{I} = \frac{1}{2}(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l. \tag{2}$$

For the 2D case the representations are similar.

The isotropic tensors $\boldsymbol{a} \in \mathbb{M}^2_s$ and $\boldsymbol{B} \in \mathbb{M}^4_s$ are represented by

$$\boldsymbol{a} = a\mathbb{1}, \qquad \boldsymbol{B} = a\mathbb{1} \otimes \mathbb{1} + b\mathbb{I}. \tag{3}$$

Consider the 3D case. The tensors

$$\boldsymbol{\Lambda}_1 = \frac{1}{3}\mathbb{1} \otimes \mathbb{1}, \qquad \boldsymbol{\Lambda}_2 = \mathbb{I} - \boldsymbol{\Lambda}_1 \tag{4}$$

are mutually orthogonal projectors having the properties

$$\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}_1, \quad \boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_1 = \boldsymbol{0}, \quad \boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_2 = \boldsymbol{\Lambda}_2. \tag{5}$$

To invert $\boldsymbol{B}$ we rearrange its representation

$$\boldsymbol{B} = (3a + b)\boldsymbol{\Lambda}_1 + b\boldsymbol{\Lambda}_2.$$

If $3a + b \neq 0$, $b \neq 0$, we find

$$\boldsymbol{B}^{-1} = (3a + b)^{-1}\boldsymbol{\Lambda}_1 + b^{-1}\boldsymbol{\Lambda}_2. \tag{6}$$

Consider the 2D case. The projectors are defined by

$$\boldsymbol{\Lambda}_1 = \frac{1}{2}\mathbb{1} \otimes \mathbb{1}, \quad \boldsymbol{\Lambda}_2 = \mathbb{I} - \boldsymbol{\Lambda}_1. \tag{7}$$

To invert $\boldsymbol{B}$ we write $\boldsymbol{B} = (2a + b)\boldsymbol{\Lambda}_1 + b\boldsymbol{\Lambda}_2$ and if $2a + b \neq 0$, $b \neq 0$, one finds

$$\boldsymbol{B}^{-1} = (2a + b)^{-1}\boldsymbol{\Lambda}_1 + b^{-1}\boldsymbol{\Lambda}_2. \tag{8}$$

2. A hole in a plane body: evaluation of change of energy by the compound asymptotics method

2.1. Setting of the problem

Let us consider a plane open domain $\Omega \subset \mathbb{R}^2$ parametrization by the Cartesian coordinate system (x_1, x_2) with the basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$. Assume that its origin $\boldsymbol{0} = (0, 0)$ lies within Ω. Let us form a family of domains ω_ε around $\boldsymbol{0}$ such that $\boldsymbol{0} \in \omega_\varepsilon$ and

$$\omega_\varepsilon = \left\{ x \,\middle|\, \frac{x}{\varepsilon} \in \omega \right\}. \tag{9}$$

Here $x = (x_1, x_2)$, ε is a small parameter and ω is an open domain in $\mathbb{R}^2$. For ε sufficiently small $\omega_\varepsilon \subset \Omega$ and the domain $\Omega_\varepsilon = \Omega \setminus \overline{\omega}_\varepsilon$ will play the role of the domain occupied by an elastic homogeneous body. The

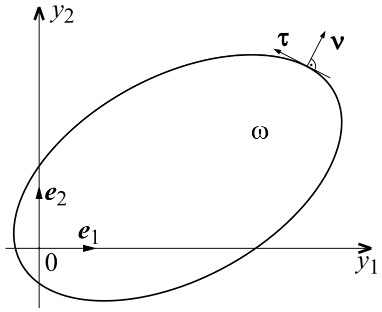

Fig. 1. The rescaled shape of the hole.

domain ω_ε represents a hole in the body. If ε changes, the holes remain homothetic to the rescaled hole ω. The domain ω and its surrounding $\mathbb{R}^2 \setminus \overline{\omega}$ will be parametrized by the Cartesian coordinates (y_1, y_2); its central point $\mathbf{0} \in \omega$ (see Fig. 1).

Let us assume that $\boldsymbol{\nu} = (\nu_1, \nu_2)$ and $\boldsymbol{\tau} = (\tau_1, \tau_2)$ are vectors: outward normal and tangent to $\partial\omega$ at a point $A = (y_1, y_2)$ (see Fig. 1). Note that $\boldsymbol{\nu}_\varepsilon$ and $\boldsymbol{\tau}_\varepsilon$ at $A_\varepsilon = (\varepsilon y_1, \varepsilon y_2) \in \partial\omega_\varepsilon$ have the same components as $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$, respectively.

We consider a plane (plane stress or plane strain) elasticity problem in the domain Ω_ε. The strains associated with a trial displacement field $\boldsymbol{u} = (u_1, u_2)$ are given by the formula

$$\epsilon_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) \tag{10}$$

and the linear constitutive relations are written in the form

$$\sigma^{\alpha\beta}(\boldsymbol{u}) = A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{u}), \tag{11}$$

where $\boldsymbol{A} = (A^{\alpha\beta\lambda\mu}) \in \mathbb{M}^4_s$ refers to the plane stress or plane strain. Tensor $\boldsymbol{A}$ is assumed to be positive definite

$$A^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu} \geqslant c\kappa_{\alpha\beta}\kappa^{\alpha\beta} \quad \forall \boldsymbol{\kappa} \in \mathbb{M}^2_s \tag{12}$$

and $c > 0$.

Assume that the external boundary $\partial\Omega$ of Ω_ε is loaded by tractions of intensity $\boldsymbol{p} = (p^\alpha)$ and that they satisfy the usual conditions of self-equilibrium

$$\int_{\partial\Omega} \boldsymbol{p} \cdot \boldsymbol{v}\, \mathrm{d}s = 0, \quad \forall \boldsymbol{v} \in \mathscr{R}, \tag{13}$$

where [1] $\mathscr{R} = \{\boldsymbol{v} | v_\alpha = v^0_\alpha + \phi e^\beta_\alpha x_\beta\}$; here v^0_α, ϕ are constants and (e^β_α) are components of the Ricci tensor; i.e. $e^1_1 = e^2_2 = 0$, $e^2_1 = -e^2_1 = 1$. The boundary $\partial\omega_\varepsilon$ is unloaded. The body forces are omitted.

The stresses associated with the unknown displacement field $\boldsymbol{u}^\varepsilon$ satisfy the homogeneous equations of equilibrium

[1] The quantity $\boldsymbol{v} = (v_1, v_2)$ represents a displacement field, while the quantity $\boldsymbol{\nu} = (\nu_1, \nu_2)$ represents a vector outward normal to $\partial\omega$. The fonts used look the same, which should not lead to misunderstandings.

$$\frac{\partial \sigma^{\alpha\beta}(\boldsymbol{u}^{\varepsilon})}{\partial x_{\beta}} = 0 \tag{14}$$

and the boundary conditions

$$\sigma^{\alpha\beta}(\boldsymbol{u}^{\varepsilon}) n_{\beta} = p^{\alpha} \quad \text{on } \partial\Omega, \tag{15}$$

$$\sigma^{\alpha\beta}(\boldsymbol{u}^{\varepsilon}) v_{\beta} = 0 \quad \text{on } \partial\omega_{\varepsilon}. \tag{16}$$

Here $\boldsymbol{n} = (n_{\beta})$, $\boldsymbol{v} = (v_{\beta})$ represent unit vectors outward normal to Ω and ω_{ε}, respectively. Due to the condition (13) the stresses $\sigma^{\alpha\beta}(\boldsymbol{u}^{\varepsilon})$ and strains $\epsilon_{\alpha\beta}(\boldsymbol{u}^{\varepsilon})$ are defined uniquely, while $\boldsymbol{u}^{\varepsilon}$ is defined up to the plane rigid motions.

The conditions (14)–(16) are equivalent to the variational equation

$$\int_{\Omega_{\varepsilon}} \sigma^{\alpha\beta}(\boldsymbol{u}^{\varepsilon}) \epsilon_{\alpha\beta}(\tilde{\boldsymbol{v}})\, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{p} \cdot \tilde{\boldsymbol{v}}\, \mathrm{d}s, \tag{17}$$

valid for all sufficiently regular $\tilde{\boldsymbol{v}}$; here $\mathrm{d}x = \mathrm{d}x_1\, \mathrm{d}x_2$.

The elastic energy stored in the body with a hole equals

$$\mathscr{E}(\Omega_{\varepsilon}) = \frac{1}{2} \int_{\Omega_{\varepsilon}} \sigma^{\alpha\beta}(\boldsymbol{u}^{\varepsilon}) \epsilon_{\alpha\beta}(\boldsymbol{u}^{\varepsilon})\, \mathrm{d}x. \tag{18}$$

The aim of the present section is to find an asymptotic formula for the change of energy $\mathscr{E}(\Omega_{\varepsilon}) - \mathscr{E}(\Omega)$, where $\mathscr{E}(\Omega)$ represents the energy stored in the body without a hole, and subjected to the same loading at $\partial\Omega$:

$$\mathscr{E}(\Omega) = \frac{1}{2} \int_{\Omega} \sigma^{\alpha\beta}(\boldsymbol{v}) \epsilon_{\alpha\beta}(\boldsymbol{v})\, \mathrm{d}x. \tag{19}$$

Here $\boldsymbol{v}$ is the displacement field within Ω satisfying:

(i) the equilibrium equation

$$\int_{\Omega} \sigma^{\alpha\beta}(\boldsymbol{v}) \epsilon_{\alpha\beta}(\tilde{\boldsymbol{v}})\, \mathrm{d}x = \int_{\partial\Omega} \boldsymbol{p} \cdot \tilde{\boldsymbol{v}}\, \mathrm{d}s \tag{20}$$

valid for all sufficiently regular $\tilde{\boldsymbol{v}}$ defined in Ω.

(ii) the constitutive relation

$$\sigma^{\alpha\beta}(\boldsymbol{v}) = A^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}(\boldsymbol{v}). \tag{21}$$

The local equations implied by (20) are

$$\frac{\partial \sigma^{\alpha\beta}(\boldsymbol{v})}{\partial x_{\beta}} = 0 \quad \text{in } \Omega, \tag{22}$$

$$\sigma^{\alpha\beta}(\boldsymbol{v}) n_{\beta} = p^{\alpha} \quad \text{on } \partial\Omega. \tag{23}$$

The energy change $\mathscr{E}(\Omega_{\varepsilon}) - \mathscr{E}(\Omega)$ can be found by the compound asymptotics method developed in Mazja et al. (1991) and in Maz'ya and Nazarov (1987). This technique is recalled in the sequel.

Let us prove now that $\mathscr{E}(\Omega_{\varepsilon}) - \mathscr{E}(\Omega) \geqslant 0$. According to the Castigliano principle we have

$$\mathscr{E}(\Omega_{\varepsilon}) = \frac{1}{2} \min \left\{ \int_{\Omega_{\varepsilon}} \tau^{\alpha\beta} C_{\alpha\beta\lambda\mu} \tau^{\lambda\mu} | \boldsymbol{\tau} = (\tau^{\alpha\beta}) \in S_{\varepsilon} \right\},$$

$$\mathscr{E}(\Omega) = \frac{1}{2} \min \left\{ \int_{\Omega} \tau^{\alpha\beta} C_{\alpha\beta\lambda\mu} \tau^{\lambda\mu} | \boldsymbol{\tau} = (\tau^{\alpha\beta}) \in S \right\}.$$

Here $\boldsymbol{C} = \boldsymbol{A}^{-1}$ and S_ε, S are the sets of statically determined stresses:

$$S_\varepsilon = \{\boldsymbol{\tau} = (\tau^{\alpha\beta}) \in L^2(\Omega_\varepsilon, \mathbb{M}_s^2) | \operatorname{div}\boldsymbol{\tau} = 0 \text{ in } \Omega_\varepsilon,\ \tau^{\alpha\beta}\nu_\beta = 0 \text{ on } \partial\omega_\varepsilon,\ \tau^{\alpha\beta}n_\beta = p^\alpha \text{ on } \partial\Omega\},$$

$$S = \{\boldsymbol{\tau} = (\tau^{\alpha\beta}) \in L^2(\Omega, \mathbb{M}_s^2) | \operatorname{div}\boldsymbol{\tau} = 0 \text{ in } \Omega,\ \tau^{\alpha\beta}n_\beta = p^\alpha \text{ on } \partial\Omega\}.$$

Let $\boldsymbol{\sigma}_{(\varepsilon)}$ and $\boldsymbol{\sigma}$ are minimizers of the above problems. Let us introduce the extended field $\tilde{\boldsymbol{\sigma}} \in S$ of the following form,

$$\tilde{\boldsymbol{\sigma}}(x) = \begin{cases} \boldsymbol{\sigma}_{(\varepsilon)}(x) & \text{if } x \in \Omega \setminus \overline{\omega}_\varepsilon, \\ 0 & \text{if } x \in \omega_\varepsilon. \end{cases}$$

Note that under some minimal regularity assumptions on the boundary of ω we have $\tilde{\boldsymbol{\sigma}} \in S$ since $\sigma_{(\varepsilon)}^{\alpha\beta}\nu_\beta = 0$ on $\partial\omega_\varepsilon$. Therefore, we have the following inequality:

$$\mathscr{E}(\Omega) \leqslant \frac{1}{2}\int_\Omega \tilde{\sigma}^{\alpha\beta}C_{\alpha\beta\lambda\mu}\tilde{\sigma}^{\lambda\mu}\,\mathrm{d}x = \frac{1}{2}\int_{\Omega_\varepsilon} \sigma_{(\varepsilon)}^{\alpha\beta}C_{\alpha\beta\lambda\mu}\sigma_{(\varepsilon)}^{\lambda\mu}\,\mathrm{d}x = \mathscr{E}(\Omega_\varepsilon).$$

2.2. Asymptotic expansions by Maz'ya and Nazarov

To determine the change of energy $\mathscr{E}(\Omega_\varepsilon) - \mathscr{E}(\Omega)$ for small ε one should apply the compound asymptotics method to disclose the way the solution $\boldsymbol{u}^\varepsilon$ depends on ε. Following Mazja et al. (1991) we represent the field $\boldsymbol{u}^\varepsilon$ in the form

$$\boldsymbol{u}^\varepsilon(x) = \boldsymbol{v}^{(0)}(x) + \varepsilon\boldsymbol{w}^{(1)}\left(\frac{x}{\varepsilon}\right) + \varepsilon^2\boldsymbol{v}^{(1)}(x) + \varepsilon^3\boldsymbol{w}^{(2)}\left(\frac{x}{\varepsilon}\right) + \cdots, \tag{24}$$

where $\boldsymbol{v}^{(0)} = \boldsymbol{v}$ is the solution for the problem posed on the domain Ω (see (20) and (21)) and the next terms introduce subsequent corrections to the boundary conditions on $\partial\omega_\varepsilon$ and $\partial\Omega$. It is assumed that

$$\frac{\partial\sigma^{\alpha\beta}(\boldsymbol{v}^{(i)}(x))}{\partial x_\beta} = 0, \quad i = 0, 1, 2, \ldots, \tag{25}$$

$$\frac{\partial\sigma_y^{\alpha\beta}(\boldsymbol{w}^{(i)}(y))}{\partial y_\beta} = 0, \quad i = 1, 2, \ldots, \tag{26}$$

where $y = (y_1, y_2)$ and

$$\sigma_y^{\alpha\beta}(\boldsymbol{u}) = A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}^y(\boldsymbol{u}), \tag{27}$$

$$\epsilon_{\lambda\mu}^y(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_\lambda}{\partial y_\mu} + \frac{\partial u_\mu}{\partial y_\lambda}\right) \tag{28}$$

for any differentiable $\boldsymbol{u} = (u_1(y), u_2(y))$.

The first approximation: $\boldsymbol{u}_{\mathrm{I}}^\varepsilon \approx \boldsymbol{v}^{(0)} = \boldsymbol{v}$ satisfies (14) and (15) but violates (16). The second approximation: $\boldsymbol{u}_{\mathrm{II}}^\varepsilon \approx \boldsymbol{v} + \varepsilon\boldsymbol{w}^{(1)}(x/\varepsilon)$ satisfies (14). The condition (16) could be satisfied with an error of order $0(\varepsilon)$, as follows. We have

$$\epsilon_{\alpha\beta}(\boldsymbol{u}_{\mathrm{II}}^\varepsilon) = \epsilon_{\alpha\beta}(\boldsymbol{v}) + \epsilon_{\alpha\beta}^y(\boldsymbol{w}^{(1)})\Big|_{y=x/\varepsilon} + 0(\varepsilon) \tag{29}$$

and hence

$$\sigma^{\alpha\beta}(\boldsymbol{u}_{\mathrm{II}}^\varepsilon)\nu_\beta = \sigma_y^{\alpha\beta}(\boldsymbol{w}^{(1)})\Big|_{y=x/\varepsilon}\nu_\beta + A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{v})(\boldsymbol{0})\nu_\beta + 0(\varepsilon). \tag{30}$$

Thus we require that

$$A^{\alpha\beta\lambda\mu}\epsilon^{y}_{\lambda\mu}(\boldsymbol{w}^{(1)})\nu_{\beta} = -A^{\alpha\beta\lambda\mu}\epsilon^{0}_{\lambda\mu}\nu_{\beta} \quad \text{on } \partial\omega, \tag{31}$$

where $\epsilon^{0}_{\lambda\mu} = \epsilon_{\lambda\mu}(\boldsymbol{v})(\boldsymbol{0})$. Note that the vector field $\boldsymbol{w}^{(1)}$ is defined in $\mathbb{R}^2 \setminus \overline{\varpi}$ and satisfies (26) and (31). This field depends linearly on the quantities $\epsilon^{0}_{\lambda\mu}$, hence $\boldsymbol{w}^{(1)}$ admits the representation

$$\boldsymbol{w}^{(1)}(y) = \epsilon^{0}_{\lambda\mu}\boldsymbol{\chi}^{(\lambda\mu)}(y), \tag{32}$$

where the functions $\boldsymbol{\chi}^{(\lambda\mu)} = \boldsymbol{\chi}^{(\mu\lambda)}$ are solutions to the boundary value problems

$$(P^{(\lambda\mu)}_{\omega})\left|\begin{array}{ll} \text{find } \boldsymbol{\chi}^{(\lambda\mu)} \text{ defined in } \mathbb{R}^2\backslash\overline{\varpi} \text{ such that} & \\ A^{\alpha\beta\gamma\delta}\frac{\partial}{\partial y_\beta}\epsilon^{y}_{\gamma\delta}(\boldsymbol{\chi}^{(\lambda\mu)}) = 0 \quad \text{in } \mathbb{R}^2\backslash\overline{\varpi}, & (33) \\ A^{\alpha\beta\gamma\delta}\epsilon^{y}_{\gamma\delta}(\boldsymbol{\chi}^{(\lambda\mu)})\nu_\beta = -A^{\alpha\beta\lambda\mu}\nu_\beta \quad \text{on } \partial\omega, & (34) \\ \boldsymbol{\chi}^{(\lambda\mu)} \to \boldsymbol{0} \quad \text{if } \|y\| \to \infty. & (35) \end{array}\right.$$

The variational formulation as well as the existence and uniqueness results of the exterior problems of linear elasticity can be found in Nazarov and Plamenevsky (1994). The related results on the polarization tensor can be found in Nazarov (2000, 2001a,b) and Argatov (1998). From these sources one can read off the mathematical details of how to construct appropriate function spaces in the domain $\mathbb{R}^2 \setminus \overline{\varpi}$ to assure the existence of $\boldsymbol{\chi}^{(\lambda\mu)}$.

However, the proof of uniqueness is much easier and is worth reporting here.

Note first, that the functions $\chi^{(\lambda\mu)}_{\alpha}$ admit the following expansion at infinity

$$\chi^{(\lambda\mu)}_{\alpha} = C^{\lambda\mu}_{\alpha\beta}\frac{y_\beta}{\|y\|} + 0(\|y\|^{-2}), \tag{36}$$

where $\|y\|^2 = (y_1)^2 + (y_2)^2$ and $C^{\lambda\mu}_{\alpha\beta}$ are constants. Let us write down the weak formulation of $P^{(\lambda\mu)}_{\omega}$. Let the circle B_R of a radius R encompass the domain ω. By multiplying (33) with $\tilde{v}_\alpha$ and integrating over $B_R \setminus \overline{\varpi}$ one finds

$$A^{\alpha\beta\lambda\mu}\int_{B_R\backslash\overline{\varpi}}\epsilon^{y}_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})\epsilon^{y}_{\alpha\beta}(\tilde{\boldsymbol{v}})\,\mathrm{d}y = A^{\alpha\beta\lambda\mu}\int_{\Gamma_R}\underline{\epsilon^{y}_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})}n_\beta\tilde{v}_\alpha\,\mathrm{d}s + A^{\alpha\beta\kappa\delta}\int_{\partial\omega}\tilde{v}_\alpha\nu_\beta\,\mathrm{d}s, \tag{37}$$

where $\Gamma_R = \partial B_R$. The term underscored above is of order $0(\|y\|^{-2})$. Thus, if $R \to \infty$ the integral over Γ_R vanishes, provided that $\tilde{v}_\alpha$ are sufficiently smooth and of bounded support. Passing to infinity: $R \to \infty$ gives the variational equation

$$A^{\alpha\beta\lambda\mu}\int_{\mathbb{R}^2\backslash\overline{\varpi}}\epsilon^{y}_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})\epsilon^{y}_{\alpha\beta}(\tilde{\boldsymbol{v}})\,\mathrm{d}y = A^{\alpha\beta\kappa\delta}\int_{\partial\omega}\tilde{v}_\alpha\nu_\beta\,\mathrm{d}s \tag{38}$$

valid for appropriately decaying trial fields $\tilde{\boldsymbol{v}}$.

Note that the right-hand side of (38) vanishes for $\tilde{\boldsymbol{v}} \in \mathscr{R}$. Let us put first $\tilde{v}_\alpha = v^0_\alpha = \text{constant}$. Then

$$\int_{\partial\omega} v^0_\alpha\nu_\beta\,\mathrm{d}s = v^0_\alpha\int_{\omega}\frac{\partial 1}{\partial y_\beta}\,\mathrm{d}y = 0. \tag{39}$$

Let us take now $\tilde{v}_\alpha = e^{\beta}_{\alpha}y_\beta$ or $\tilde{v}_1 = y_2$, $\tilde{v}_2 = -y_1$. The right-hand side of (38) equals

$$A^{\alpha\beta\kappa\delta}e^{\sigma}_{\alpha}\int_{\partial\omega}y_\sigma\nu_\beta\,\mathrm{d}s = A^{\alpha\beta\kappa\delta}e^{\sigma}_{\alpha}\int_{\partial\omega}y_\sigma\frac{\partial y_\beta}{\partial y_\lambda}\nu_\lambda\,\mathrm{d}s. \tag{40}$$

Since

$$\sum_{\lambda}\frac{\partial^2 y_\beta}{\partial y_\lambda\partial y_\lambda}y_\sigma = 0$$

we have

$$\sum_{\lambda}\left[\frac{\partial}{\partial y_\lambda}\left(\frac{\partial y_\beta}{\partial y_\lambda}y_\sigma\right)-\frac{\partial y_\beta}{\partial y_\lambda}\frac{\partial y_\sigma}{\partial y_\lambda}\right]=0.$$

Integrating over ω one finds

$$\int_{\partial\omega}\frac{\partial y_\beta}{\partial y_\lambda}y_\sigma \nu_\lambda\,\mathrm{d}s=\sum_{\lambda}\int_{\omega}\frac{\partial y_\beta}{\partial y_\lambda}\frac{\partial y_\sigma}{\partial y_\lambda}\,\mathrm{d}y. \tag{41}$$

Hence we find a useful identity

$$\int_{\partial\omega} y_\sigma \nu_\beta\,\mathrm{d}s=\delta_{\sigma\beta}|\omega|, \tag{42}$$

where $|\omega|$ represents the area of ω. Due to symmetry of $(A^{\alpha\beta\kappa\delta})$ with respect to the first two indices one notes that the expression (40) vanishes. Since the right-hand side of (38) vanishes for all $\tilde{\boldsymbol{v}}\in\mathscr{R}$, the fields $\boldsymbol{\chi}^{(\kappa\delta)}$ are uniquely determined.

Let us define the auxiliary vector fields

$$\boldsymbol{E}^{(\lambda\mu)}(y)=\frac{1}{2}(y_\lambda\boldsymbol{e}_\mu+y_\mu\boldsymbol{e}_\lambda),\quad \lambda,\mu\in\{1,2\}, \tag{43}$$

with components

$$\boldsymbol{E}^{(\lambda\mu)}_\alpha(y)=\frac{1}{2}(y_\lambda\delta_{\mu\alpha}+y_\mu\delta_{\lambda\alpha}). \tag{44}$$

The strains associated with these fields form a unit tensor (2) in $\mathbb{M}^4_s$

$$\epsilon^y_{\alpha\beta}(\boldsymbol{E}^{(\lambda\mu)})=\mathbb{I}^{\lambda\mu}_{\alpha\beta}. \tag{45}$$

Here

$$\mathbb{I}^{\lambda\mu}_{\alpha\beta}=\frac{1}{2}(\delta^\lambda_\alpha\delta^\mu_\beta+\delta^\mu_\alpha\delta^\lambda_\beta). \tag{46}$$

By virtue of (45) one can rewrite the conditions (34) as follows:

$$A^{\alpha\beta\lambda\mu}\epsilon^y_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)}+\boldsymbol{E}^{(\kappa\delta)})\nu_\beta=0. \tag{47}$$

Let us introduce new fields

$$\boldsymbol{\psi}^{(\alpha\beta)}(y)=\boldsymbol{\chi}^{(\alpha\beta)}(y)+\boldsymbol{E}^{(\alpha\beta)}(y). \tag{48}$$

By (47) we see that these fields are solutions to the following problems

$$(\widetilde{P}^{\kappa\delta}_\omega)\left|\begin{array}{ll}\text{find } \boldsymbol{\psi}^{(\kappa\delta)} \text{ defined in } \mathbb{R}^2\backslash\overline{\omega} \text{ such that} & \\ A^{\alpha\beta\lambda\mu}\frac{\partial}{\partial y_\beta}\epsilon^y_{\lambda\mu}(\boldsymbol{\psi}^{(\kappa\delta)})=0 \quad \text{in } \mathbb{R}^2\backslash\overline{\omega}, & (49)\\ A^{\alpha\beta\lambda\mu}\epsilon^y_{\lambda\mu}(\boldsymbol{\psi}^{(\kappa\delta)})\nu_\beta=0 \quad \text{on } \partial\omega, & (50)\\ \boldsymbol{\psi}^{(\kappa\delta)}\to\boldsymbol{E}^{(\kappa\delta)} \quad \text{if } \|y\|\to\infty. & (51)\end{array}\right.$$

Let us come back to the asymptotic expansion (24). Substitution of (32) gives

$$\boldsymbol{u}^\varepsilon(x)=\boldsymbol{v}(x)+\varepsilon\epsilon^0_{\lambda\mu}\boldsymbol{\chi}^{(\lambda\mu)}\left(\frac{x}{\varepsilon}\right)+\varepsilon^2\boldsymbol{v}^{(1)}(x)+\cdots \tag{52}$$

In the vicinity of the boundary $\partial\omega_\varepsilon$ the function $\boldsymbol{v}(x)$ has the following expansion:

$$\boldsymbol{v}(x) = \underline{\boldsymbol{v}(\boldsymbol{0})} + \epsilon^0_{\lambda\mu}\boldsymbol{E}^{(\lambda\mu)}(x) + \underline{\phi\boldsymbol{r}(x)} + 0(\varepsilon^2), \tag{53}$$

where $\boldsymbol{r}(x) = (1/2)(-x_2\boldsymbol{e}_1 + x_1\boldsymbol{e}_2)$ and ϕ represents an angle of a rigid rotation in the (x_1, x_2) plane. The terms underscored in (53) belong to the set $\mathscr{R}$. Hence the strains are

$$\epsilon_{\alpha\beta}(\boldsymbol{v}) = \epsilon^0_{\alpha\beta} + 0(\varepsilon) \tag{54}$$

and

$$\epsilon_{\alpha\beta}(\boldsymbol{u}^\varepsilon) = \epsilon^0_{\alpha\beta} + \epsilon^0_{\lambda\mu}\epsilon^y_{\alpha\beta}(\boldsymbol{\chi}^{(\lambda\mu)})\Big|_{y=x/\varepsilon} + 0(\varepsilon) \tag{55}$$

or

$$\epsilon_{\alpha\beta}(\boldsymbol{u}^\varepsilon) = \epsilon^0_{\lambda\mu}\epsilon^y_{\alpha\beta}(\boldsymbol{\psi}^{(\lambda\mu)})\Big|_{y=x/\varepsilon} + 0(\varepsilon). \tag{56}$$

In the further use of the formulae above a deeper insight into the behavior of the functions $\chi^{(\lambda\mu)}_\alpha$ is necessary. This will be the subject of the subsequent section.

2.3. Tensor of Polya–Szegö

The solutions of the problems $P^{(\lambda\mu)}_\omega$ have the following expansions around $y = \infty$ (see Mazja et al., 1991)

$$\chi^{(\kappa\delta)}_\sigma(y) = M^{\kappa\delta\lambda\mu}\epsilon^y_{\lambda\mu}(\boldsymbol{T}_{(\sigma)}(y)) + 0(\|y\|^{-2}), \tag{57}$$

where the coefficients $(M^{\kappa\delta\lambda\mu})$ are constant and the vector fields $\boldsymbol{T}_{(1)}$, $\boldsymbol{T}_{(2)}$ represent the displacement fields of an infinite body subjected to the point loads $\boldsymbol{b}_{(1)} = \delta(y)\boldsymbol{e}_1$, $\boldsymbol{b}_{(2)} = \delta(y)\boldsymbol{e}_2$, respectively. The point loads are concentrated at the origin $y = \boldsymbol{0}$. Thus the components $(\boldsymbol{T}_{(\sigma)})$ form the Somigliana or Kelvin solutions. They satisfy the Lame equations

$$A^{\alpha\beta\iota\varrho}\frac{\partial^2(\boldsymbol{T}_{(\sigma)})_\iota}{\partial y_\varrho \partial y_\beta} + \delta_{\alpha\sigma}\delta(y - \boldsymbol{0}) = 0. \tag{58}$$

The fields $(\boldsymbol{T}_{(\sigma)})_\iota$ can be found by performing the Fourier transform of Eq. (58). It occurs that these fields include singularities of order $0(\ln\|y\|)$, while the fields

$$U_{(\alpha\beta)\sigma} = \epsilon^y_{\alpha\beta}(\boldsymbol{T}_{(\sigma)}), \tag{59}$$

include singularities of order $0(\|y\|^{-1})$.

The tensor of elastic moduli of an isotropic body in the 2D problem (both: the plane strain and plane stress) is represented by

$$\boldsymbol{A} = 2k\boldsymbol{\Lambda}_1 + 2\mu\boldsymbol{\Lambda}_2, \tag{60}$$

where k and μ are bulk and shear moduli and the operators $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$ are given by (7). The components of the Somigliana solutions are expressed by (see Hahn, 1985, p. 274)

$$(\boldsymbol{T}_{(\sigma)}(y))_\alpha = \frac{1}{4\pi\mu(k+\mu)}\left[-(k+2\mu)\delta_{\sigma\alpha}\ln\|y\| + k\frac{y_\sigma y_\alpha}{\|y\|^2}\right]. \tag{61}$$

The components $(M^{\kappa\delta\lambda\mu})$ entering (57) form the Polya–Szegö tensor, similar to that introduced in the potential theory (see Schiffer and Szegö, 1949; Schiffer, 1956). The notion: "Polya–Szegö tensor" is used in Mazja et al. (1991) and Movchan and Movchan (1995, Section 5.1).

Since the fields $\chi_\alpha^{\lambda\mu}$ are of length dimension the components $(M^{\kappa\delta\lambda\mu})$ are of force dimension (N). These components depend on the shape of ω and on the moduli k and μ. Thus there exists a non-dimensional tensor $\widetilde{\boldsymbol{M}}$ such that

$$\boldsymbol{M} = |\omega| k \widetilde{\boldsymbol{M}}. \tag{62}$$

Knowing the fields $\boldsymbol{\chi}^{(\kappa\delta)}$ and the Somigliana vectors $\boldsymbol{T}_{(\sigma)}$ and imposing the following symmetry properties

$$M^{\kappa\delta\lambda\mu} = M^{\lambda\mu\kappa\delta}, \quad M^{\kappa\delta\lambda\mu} = M^{\delta\kappa\lambda\mu}, \quad M^{\kappa\delta\lambda\mu} = M^{\kappa\delta\mu\lambda} \tag{63}$$

one can determine the components $(M^{\kappa\delta\lambda\mu})$ directly from the representation (57). Alternatively, these components are expressed by the following integral formulae

$$\text{(i)} \quad M^{\kappa\delta\sigma\gamma} = -A^{\alpha\beta\kappa\delta} \int_{\partial\omega} \psi_\alpha^{(\sigma\gamma)} v_\beta \, \mathrm{d}s, \tag{64}$$

$$\text{(ii)} \quad M^{\kappa\delta\sigma\gamma} = -A^{\kappa\delta\sigma\gamma} |\omega| - \mathscr{M}^{\kappa\delta\sigma\gamma}, \tag{65}$$

where

$$\mathscr{M}^{\kappa\delta\sigma\gamma} = A^{\alpha\beta\lambda\mu} \int_{\mathbb{R}^2 \setminus \overline{\omega}} \epsilon^y_{\lambda\mu}(\chi^{(\kappa\delta)}) \epsilon^y_{\alpha\beta}(\chi^{(\sigma\gamma)}) \, \mathrm{d}y \tag{66}$$

or

$$\mathscr{M}^{\kappa\delta\sigma\gamma} = A^{\alpha\beta\kappa\delta} \int_{\partial\omega} \chi_\alpha^{(\sigma\gamma)} v_\beta \, \mathrm{d}s. \tag{67}$$

Tensor $\mathscr{M}$ is analogous to the *mass matrix tensor* of Polya and Szegö (see Schiffer and Szegö, 1949).

In a different, vectorial notation, according to which the tensors $\boldsymbol{M}$ and $\mathscr{M}$ are of the rank two and of dimension 6×6, the formulae (64)–(67) have been derived in Mazja et al. (1991). Due to some technical reasons, in order to emphasize new subtle arguments, it is thought appropriate to give below the complete derivation of the formulae (64)–(67).

Proof of (i). Let us encompass the domain ω by a circle B_R (see Section 2.2). The fields $\boldsymbol{\psi}^{(\sigma\gamma)}$ satisfy an equation of the form similar to (37)

$$A^{\alpha\beta\lambda\mu} \int_{B_R \setminus \overline{\omega}} \epsilon^y_{\lambda\mu}(\boldsymbol{\psi}^{(\sigma\gamma)}) \epsilon^y_{\alpha\beta}(\hat{\boldsymbol{v}}) \, \mathrm{d}y = A^{\alpha\beta\lambda\mu} \int_{\Gamma_R} \epsilon^y_{\lambda\mu}(\boldsymbol{\psi}^{(\sigma\gamma)}) n_\beta \hat{v}_\alpha \, \mathrm{d}s \tag{68}$$

valid for sufficiently regular fields $\hat{v}_\alpha$. The boundary conditions (50) have been taken into account.

Let us put $\tilde{\boldsymbol{v}} = \boldsymbol{\psi}^{(\sigma\gamma)}$ into (37) and $\hat{\boldsymbol{v}} = \boldsymbol{\chi}^{(\kappa\delta)}$ into (68). Thus we arrive at two identities of the same left-hand sides. Equating the right-hand sides gives the formula

$$N^{\kappa\delta\sigma\gamma} = -A^{\alpha\beta\kappa\delta} \int_{\partial\omega} \psi_\alpha^{(\sigma\gamma)} v_\beta \, \mathrm{d}s, \tag{69}$$

where

$$N^{\kappa\delta\sigma\gamma} = A^{\alpha\beta\lambda\mu} \int_{\Gamma_R} \left[\epsilon^y_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)}) \psi_\alpha^{(\sigma\gamma)} - \epsilon^y_{\lambda\mu}(\boldsymbol{\psi}^{(\sigma\gamma)}) \chi_\alpha^{(\kappa\delta)} \right] n_\beta \, \mathrm{d}s. \tag{70}$$

Thus the tensor $\boldsymbol{N}$ does not depend on R. The components $N^{\kappa\delta\sigma\gamma}$ will be found by passing with R to infinity in the formula (70). By (48) and (57) we note that

$$\psi_\alpha^{(\sigma\gamma)}\big|_{\Gamma_R} = E_\alpha^{(\sigma\gamma)} + 0(R^{-1}), \tag{71}$$

$$\epsilon^{y}_{\lambda\mu}(\boldsymbol{\psi}^{(\sigma\gamma)})\big|_{\Gamma_R} = \mathbb{I}^{\sigma\gamma}_{\lambda\mu} + 0(R^{-2}), \tag{72}$$

and substitute these expressions into (70). By disregarding the terms that do not contribute to the final result, we obtain

$$N^{\kappa\delta\sigma\gamma} = \lim_{R\to\infty} A^{\alpha\beta\lambda\mu} \int_{\Gamma_R} \left[\epsilon^{y}_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})E^{(\sigma\gamma)}_{\alpha} - \chi^{\kappa\delta}_{\alpha}\mathbb{I}^{\sigma\gamma}_{\lambda\mu}\right] n_\beta \,\mathrm{d}s \tag{73}$$

and substitution of the representation (57) gives

$$N^{\kappa\delta\sigma\gamma} = M^{\kappa\delta\iota\varrho}\eta^{\sigma\gamma}_{\iota\varrho}, \tag{74}$$

$$\eta^{\sigma\gamma}_{\iota\varrho} = A^{\alpha\beta\lambda\mu} \lim_{R\to\infty} \int_{\Gamma_R} \left[\epsilon^{y}_{\lambda\mu}(\boldsymbol{U}_{(\iota\varrho)})E^{(\sigma\gamma)}_{\alpha} - U_{(\iota\varrho)\alpha}\mathbb{I}^{\sigma\gamma}_{\lambda\mu}\right] n_\beta \,\mathrm{d}s. \tag{75}$$

Here

$$(\boldsymbol{U}_{(\iota\varrho)}) = \sum_{\sigma} U_{(\iota\varrho)\sigma}\boldsymbol{e}_\sigma \tag{76}$$

and $U_{(\iota\varrho)1}$, $U_{(\iota\varrho)2}$ are defined by (59). Our aim is to show that

$$\eta^{\sigma\gamma}_{\iota\varrho} = \mathbb{I}^{\sigma\gamma}_{\iota\varrho}. \tag{77}$$

The key idea of the proof is to make use of the variational counterpart of the Somigliana equation (58)

$$\int_{B_R} A^{\alpha\beta\iota\varrho}U_{(\iota\varrho)\sigma}\epsilon^{y}_{\alpha\beta}(\boldsymbol{\varphi})\,\mathrm{d}y = \boldsymbol{\varphi}(\mathbf{0})\cdot\boldsymbol{e}_\sigma + \int_{\Gamma_R} A^{\alpha\beta\iota\varrho}U_{(\iota\varrho)\sigma}n_\beta\varphi_\alpha\,\mathrm{d}s. \tag{78}$$

Let us remind that the components of the point loads $\boldsymbol{b}_{(\alpha)}$ are $\boldsymbol{b}_{(\alpha)\lambda} = \delta_{\alpha\lambda}\delta(y)$. The hyperforce

$$\boldsymbol{s}^{(\lambda\mu)} = \sum_{\alpha} \epsilon^{y}_{\lambda\mu}(\boldsymbol{b}_{(\alpha)})\boldsymbol{e}_\alpha \tag{79}$$

is applied at the point $\mathbf{0} = (0,0)$; the indices λ and μ are viewed here as fixed. The components of $\boldsymbol{s}^{(\lambda\mu)}$ are given by

$$s^{(\lambda\mu)}_{\alpha} = \frac{1}{2}\left[\frac{\partial}{\partial y_\mu}(\delta_{\alpha\lambda}\delta(y)) + \frac{\partial}{\partial y_\lambda}(\delta_{\alpha\mu}\delta(y))\right]$$

or

$$s^{(\lambda\mu)}_{\alpha} = \frac{1}{2}\left(\delta_{\alpha\lambda}\frac{\partial\delta}{\partial y_\mu} + \delta_{\alpha\mu}\frac{\partial\delta}{\partial y_\lambda}\right). \tag{80}$$

We note that the force $\boldsymbol{s}^{(\lambda\mu)}$ brings about the displacement field $\boldsymbol{U}_{(\lambda\mu)}$ given by (76) and (59). This displacement field satisfies the variational equation (78) in which the force $\boldsymbol{b}_{(\sigma)}$ is now replaced with $\boldsymbol{s}^{(\lambda\mu)}$. It reads

$$\int_{B_R} A^{\alpha\beta\iota\varrho}\epsilon^{y}_{\iota\varrho}(\boldsymbol{U}_{(\lambda\mu)})\epsilon^{y}_{\alpha\beta}(\boldsymbol{\varphi})\,\mathrm{d}y = \langle\boldsymbol{s}^{(\lambda\mu)}, \boldsymbol{\varphi}\rangle + \int_{\Gamma_R} A^{\alpha\beta\iota\varrho}\epsilon^{y}_{\iota\varrho}(\boldsymbol{U}_{(\lambda\mu)})n_\beta\varphi_\alpha\,\mathrm{d}s, \tag{81}$$

where φ_α denotes the components of the vector test function $\boldsymbol{\varphi}$.

Let us define the mollifier

$$\phi_h(y) = h^{-2}\phi\left(\frac{y}{h}\right), \tag{82}$$

where h is a small positive number and

$$\phi(x) = \begin{cases} C\exp[-1/(1-\|x\|^2)] & \text{if } \|x\| < 1, \\ 0 & \text{if } \|x\| \geqslant 1. \end{cases} \tag{83}$$

Constant C is chosen such that

$$\int_{\mathbb{R}^2} \phi(x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}^2} \phi_h(y)\,\mathrm{d}y = 1. \tag{84}$$

Let us introduce the pseudoforces $\boldsymbol{s}_h^{(\lambda\mu)}$ of components $(\boldsymbol{s}_h^{(\lambda\mu)})_\alpha$ given by (80) with $\delta(y)$ replaced by $\phi_h(y)$. These pseudoforces bring about the displacement fields $\boldsymbol{U}^h_{(\lambda\mu)}$ of components $(\boldsymbol{U}^h_{(\lambda\mu)})_\sigma = \epsilon^y_{\lambda\mu}(\boldsymbol{T}^h_{(\sigma)})$ which satisfy the variational equation

$$A^{\alpha\beta\iota\varrho}\int_{B_R} \epsilon^y_{\iota\varrho}(\boldsymbol{U}^h_{(\lambda\mu)})\epsilon^y_{\alpha\beta}(\boldsymbol{\varphi})\,\mathrm{d}y = \int_{B_R} \boldsymbol{\varphi}\cdot\boldsymbol{s}_h^{(\lambda\mu)}\,\mathrm{d}y + A^{\alpha\beta\iota\varrho}\int_{\Gamma_R} \epsilon^y_{\iota\varrho}(\boldsymbol{U}^h_{(\lambda\mu)})n_\beta\varphi_\alpha\,\mathrm{d}s \tag{85}$$

for sufficiently regular $\boldsymbol{\varphi}$. Let us substitute $\boldsymbol{\varphi} = \boldsymbol{E}^{(\kappa\delta)}$ to obtain

$$A^{\alpha\beta\iota\varrho}\int_{B_R} \epsilon^y_{\iota\varrho}(\boldsymbol{U}^h_{(\lambda\mu)})\,\mathrm{d}y = \int_{B_R} \boldsymbol{E}^{(\kappa\delta)}\cdot\boldsymbol{s}_h^{(\lambda\mu)}\,\mathrm{d}y + A^{\alpha\beta\iota\varrho}\int_{\Gamma_R} \epsilon^y_{\iota\varrho}(\boldsymbol{U}^h_{(\lambda\mu)})E^{(\kappa\delta)}_\alpha n_\beta\,\mathrm{d}s. \tag{86}$$

Note that the stresses $\sigma^{\alpha\beta}_y(\boldsymbol{E}^{(\kappa\delta)})$ satisfy the homogeneous equations of equilibrium. Thus the following variational equation holds

$$A^{\alpha\beta\iota\varrho}\int_{B_R} \epsilon^y_{\iota\varrho}(\boldsymbol{E}^{(\kappa\delta)})\epsilon^y_{\alpha\beta}(\tilde{\boldsymbol{v}})\,\mathrm{d}y = A^{\alpha\beta\iota\varrho}\int_{\Gamma_R} \epsilon^y_{\iota\varrho}(\boldsymbol{E}^{(\kappa\delta)})n_\beta\tilde{v}_\alpha\,\mathrm{d}s \tag{87}$$

for sufficiently regular $\tilde{v}_\alpha$. Let us substitute $\tilde{\boldsymbol{v}} = \boldsymbol{U}^h_{(\lambda\mu)}$ to obtain

$$A^{\alpha\beta\kappa\delta}\int_{B_R} \epsilon^y_{\alpha\beta}(\boldsymbol{U}^h_{(\lambda\mu)})\,\mathrm{d}y = A^{\alpha\beta\kappa\delta}\int_{\Gamma_R} U^h_{(\lambda\mu)\alpha}n_\beta\,\mathrm{d}s. \tag{88}$$

By equating the right-hand sides of (86) and (88) one finds

$$A^{\alpha\beta\iota\varrho}\int_{\Gamma_R}\left[\epsilon^y_{\iota\varrho}(\boldsymbol{U}^h_{(\lambda\mu)})E^{(\kappa\delta)}_\alpha - \boldsymbol{U}^h_{(\lambda\mu)\alpha}\mathbb{I}^{\kappa\delta}_{\iota\varrho}\right]n_\beta\,\mathrm{d}s = -\int_{B_R}\boldsymbol{E}^{(\kappa\delta)}\cdot\boldsymbol{s}_h^{(\lambda\mu)}\,\mathrm{d}y \tag{89}$$

Let us pass with h to zero. The left-hand side of (89) tends to $\eta^{\kappa\delta}_{\lambda\mu}$ defined by (75). Let us find the product: $\boldsymbol{E}^{(\kappa\delta)}\cdot\boldsymbol{s}_h^{(\lambda\mu)}$. To this end let us compute

$$\boldsymbol{\varphi}\cdot\boldsymbol{s}_h^{(\lambda\mu)} = \varphi_\alpha(\boldsymbol{s}_h^{(\lambda\mu)})_\alpha = \frac{1}{2}\left(\varphi_\lambda\frac{\partial\phi_h}{\partial y_\mu} + \varphi_\mu\frac{\partial\phi_h}{\partial y_\lambda}\right) = \epsilon^y_{\lambda\mu}(\phi_h\boldsymbol{\varphi}) - \epsilon^y_{\lambda\mu}(\boldsymbol{\varphi})\phi_h(y).$$

Hence

$$\int_{B_R}\boldsymbol{\varphi}\cdot\boldsymbol{s}_h^{(\lambda\mu)}\,\mathrm{d}y = \int_{B_R}\epsilon^y_{\lambda\mu}(\phi_h\boldsymbol{\varphi})\,\mathrm{d}y - \int_{B_R}\epsilon^y_{\lambda\mu}(\boldsymbol{\varphi})\phi_h(y)\,\mathrm{d}y \to -\epsilon^y_{\lambda\mu}(\boldsymbol{\varphi})(\mathbf{0}) \tag{90}$$

if $h \searrow 0$. Therefore, the right-hand side of (89) tends to $\epsilon^y_{\lambda\mu}(\boldsymbol{E}^{(\kappa\delta)}) = \mathbb{I}^{\kappa\delta}_{\lambda\mu}$, which proves (77). Substitution of (77) into (74) gives $\boldsymbol{N} = \boldsymbol{M}$, with $\boldsymbol{N}$ being defined by (69). The thesis (i) or the formula (64) is proved. □

Proof of (ii). We substitute $\tilde{\boldsymbol{v}} = \boldsymbol{\chi}^{(\sigma\gamma)}$ into the variational equation (37). We note that

$$\int_{\Gamma_R}\epsilon^y_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)})\chi^{(\sigma\gamma)}_\alpha n_\beta\,\mathrm{d}s \to 0 \tag{91}$$

when $R \to 0$. Indeed, $\epsilon^y_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)}) = 0(R^{-2})$ and $\chi^{(\sigma\gamma)}_\alpha = (R^{-1})$, $\mathrm{d}s = R\,\mathrm{d}\theta$. Thus the above result holds. Consequently, one finds

$$A^{\alpha\beta\lambda\mu} \int_{\mathbb{R}^2 \setminus \overline{\omega}} \epsilon^y_{\lambda\mu}(\boldsymbol{\chi}^{(\kappa\delta)}) \epsilon^y_{\alpha\beta}(\chi^{(\sigma\gamma)})\,\mathrm{d}y = A^{\alpha\beta\kappa\delta} \int_{\partial\omega} \chi^{(\sigma\gamma)}_\alpha \nu_\beta\,\mathrm{d}s \tag{92}$$

which proves that both the derivations (66) and (67) of the tensor $\mathscr{M}$ are equivalent.

Let us substitute (48) into (64) and make use of notation (67). One obtains

$$M^{\kappa\delta\sigma\gamma} = -A^{\alpha\beta\kappa\delta} \int_{\partial\omega} E^{(\sigma\gamma)}_\alpha \nu_\beta\,\mathrm{d}s - \mathscr{M}^{\kappa\delta\sigma\gamma}. \tag{93}$$

By using (44) and (42) one computes

$$\begin{aligned} \int_{\partial\omega} E^{(\sigma\gamma)}_\alpha \nu_\beta\,\mathrm{d}s &= \frac{1}{2}\int_{\partial\omega} (y_\gamma\delta_{\alpha\sigma} + y_\sigma\delta_{\alpha\gamma})\nu_\beta\,\mathrm{d}s = \frac{1}{2}\delta_{\alpha\sigma}\int_{\partial\omega} y_\gamma\nu_\beta\,\mathrm{d}s + \frac{1}{2}\delta_{\alpha\gamma}\int_{\partial\omega} y_\sigma\nu_\beta\,\mathrm{d}s \\ &= \frac{1}{2}\delta_{\alpha\sigma}\delta_{\gamma\beta}|\omega| + \frac{1}{2}\delta_{\alpha\gamma}\delta_{\sigma\beta}|\omega| = \mathbb{I}^{\sigma\gamma}_{\alpha\beta}|\omega| \end{aligned} \tag{94}$$

which confirms (65). The thesis (ii) is proved. □

The symmetry properties (63) are direct consequences of the formula (65). Let us consider a quadratic form

$$f(\boldsymbol{q}) = q_{\alpha\beta}(-M^{\alpha\beta\lambda\mu})q_{\lambda\mu}, \quad \boldsymbol{q} \in \mathbb{M}^2_s. \tag{95}$$

By (65) and (66) we have

$$f(\boldsymbol{q}) = q_{\kappa\delta}A^{\kappa\delta\sigma\gamma}q_{\sigma\gamma}|\omega| + \int_{\mathbb{R}^2 \setminus \overline{\omega}} \epsilon^y_{\lambda\mu}(\boldsymbol{\phi})A^{\lambda\mu\alpha\beta}\epsilon^y_{\alpha\beta}(\boldsymbol{\phi})\,\mathrm{d}y \tag{96}$$

with $\boldsymbol{\phi} = q_{\alpha\beta}\boldsymbol{\chi}^{(\alpha\beta)}$. Since $\boldsymbol{\chi}^{(\alpha\beta)} \notin \mathscr{R}$ we know that $\epsilon^y_{\lambda\mu}(\boldsymbol{q}) \neq 0$. The estimate (12) implies $f(\boldsymbol{q}) > 0$ if $\boldsymbol{q} \neq \boldsymbol{0}$. Thus the tensor $\boldsymbol{M}$ is negative definite.

The role of the tensor $\boldsymbol{M}$ will be cleared up in the subsequent section.

2.4. Change of energy according to Mazja et al.

The compound asymptotics method makes it possible to solve the problems of perturbation of a large class of shape functionals (see Nazarov and Sokołowski, 2003). Change of the energy functional for the Neumann problem, brought about by drilling of a small cavity, was first determined in Maz'ya and Nazarov (1987). The similar perturbation problem for the linear elasticity was considered in Mazja et al. (1991). However, the final result is not reported in this book. This final result is recalled by Movchan and Movchan (1995, Section 5.1.3), where a reference is made to an unavailable paper by Zorin et al. (1989). This final result reads

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) - \frac{1}{2}\varepsilon^2\epsilon^0_{\alpha\beta}M^{\alpha\beta\lambda\mu}\epsilon^0_{\lambda\mu} + \mathrm{o}(\varepsilon^2), \tag{97}$$

where $\mathscr{E}(\Omega_\varepsilon)$ and $\mathscr{E}(\Omega)$ have been defined by (18) and (19) and represent the elastic energies stored in the body with a hole and without a hole, respectively. Moreover, $\epsilon^0_{\alpha\beta} = \epsilon_{\alpha\beta}(\boldsymbol{v})(\boldsymbol{0})$ represent strains measured in a body without a hole at the point where the hole starts to nucleate.

Since the derivation of the formula (97) is still unpublished in the available literature, it is thought appropriate to present it in detail. By (17) and (18) the energy stored in the body with a hole equals

$$\mathscr{E}(\Omega_\varepsilon) = \frac{1}{2}\int_{\partial\Omega} \boldsymbol{p}\cdot\boldsymbol{u}^\varepsilon\,\mathrm{d}s. \tag{98}$$

Substitution of (24) into (98) must be done with care. By (57) we have

$$\chi_{\sigma}^{(\kappa\delta)}\left(\frac{x}{\varepsilon}\right) = \varepsilon M^{\kappa\delta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{T}_{(\sigma)}(x)) + \varepsilon^2 0(\|x\|^{-2}). \tag{99}$$

Substitution into (24), with making use of (32) gives

$$\boldsymbol{u}^{\varepsilon}(x) = \boldsymbol{v}(x) + \underline{\varepsilon^2\tilde{\boldsymbol{u}}(x)} + \varepsilon^3 0(\|x\|^{-2}) + \mathrm{o}(\varepsilon^2) \tag{100}$$

with

$$\tilde{\boldsymbol{u}}(x) = \epsilon^0_{\lambda\mu}M^{\lambda\mu\gamma\delta}\boldsymbol{U}^x_{(\gamma\delta)}(x) + \boldsymbol{v}^{(1)}(x). \tag{101}$$

Here

$$\boldsymbol{U}^x_{(\gamma\delta)} = \left[\epsilon_{\gamma\delta}(\boldsymbol{T}_{(1)}), \epsilon_{\gamma\delta}(\boldsymbol{T}_{(2)})\right].$$

The term underscored in (100) must satisfy the homogeneous boundary condition

$$\sigma^{\alpha\beta}(\tilde{\boldsymbol{u}})n_\beta = 0 \quad \text{on } \partial\Omega. \tag{102}$$

Moreover, by (25) we have

$$\frac{\partial}{\partial x_\beta}\sigma^{\alpha\beta}(\boldsymbol{v}^{(1)}) = 0 \quad \text{in } \Omega. \tag{103}$$

Consequently, the function $\boldsymbol{v}^{(1)}$ satisfies (103) and the non-homogeneous boundary condition on $\partial\Omega$

$$\sigma^{\alpha\beta}(\boldsymbol{v}^{(1)})n_\beta = -\epsilon^0_{\lambda\mu}M^{\lambda\mu\gamma\delta}\sigma^{\alpha\beta}(\boldsymbol{U}^x_{(\gamma\delta)})n_\beta \quad \text{on } \partial\Omega. \tag{104}$$

By analogy with the variational equation (81) one finds the variational equation for $\boldsymbol{U}^x_{(\gamma\delta)}$

$$A^{\alpha\beta\iota\varrho}\int_\Omega \epsilon_{\iota\varrho}(\boldsymbol{U}^x_{(\gamma\delta)})\epsilon_{\alpha\beta}(\boldsymbol{\varphi})\,\mathrm{d}x = \langle \boldsymbol{s}^{(\gamma\delta)}, \boldsymbol{\varphi}\rangle + \int_{\partial\Omega}\sigma^{\alpha\beta}(\boldsymbol{U}^x_{(\gamma\delta)})n_\beta\varphi_\alpha\,\mathrm{d}s, \tag{105}$$

where $\boldsymbol{\varphi}$ is an appropriately regular function defined in $\mathbb{R}^2$ and (see (90))

$$\langle \boldsymbol{s}^{(\gamma\delta)}, \boldsymbol{\varphi}\rangle = \lim_{h\searrow 0}\langle \boldsymbol{s}_h^{(\gamma\delta)}, \boldsymbol{\varphi}\rangle = -\epsilon_{\gamma\delta}(\boldsymbol{\varphi})(\mathbf{0}). \tag{106}$$

By (103) and (104) we note that $\boldsymbol{v}^{(1)}$ admits the following representation

$$\boldsymbol{v}^{(1)} = M^{\lambda\mu\gamma\delta}\epsilon^0_{\lambda\mu}\boldsymbol{z}_{(\gamma\delta)}(x), \tag{107}$$

where the functions $\boldsymbol{z}_{(\gamma\delta)} = \boldsymbol{z}_{(\delta\gamma)}$ satisfy

$$(\widetilde{P}^{\gamma\delta}_{1\omega})\left|\begin{array}{ll} \dfrac{\partial}{\partial x_\beta}\sigma^{\alpha\beta}(\boldsymbol{z}_{(\gamma\delta)}) = 0 \quad \text{in } \Omega, & (108)\\ \sigma^{\alpha\beta}(\boldsymbol{z}_{(\gamma\delta)})n_\beta = -\sigma^{\alpha\beta}(\boldsymbol{U}^x_{(\gamma\delta)})n_\beta \quad \text{on } \partial\Omega & (109)\end{array}\right.$$

and sigma $\sigma^{\alpha\beta}(\cdot)$ are given by (11).

Substitution of (107) into (101) gives

$$\tilde{\boldsymbol{u}} = \epsilon^0_{\lambda\mu}M^{\lambda\mu\gamma\delta}(\boldsymbol{U}^x_{(\gamma\delta)} + \boldsymbol{z}_{(\gamma\delta)}). \tag{110}$$

The functions $\boldsymbol{z}_{(\gamma\delta)}$ satisfy the following variational equation

$$\int_\Omega A^{\alpha\beta\iota\varrho}\epsilon_{\iota\varrho}(\boldsymbol{z}_{(\gamma\delta)})\epsilon_{\alpha\beta}(\boldsymbol{\varphi})\,\mathrm{d}x = -\int_{\partial\Omega}\sigma^{\alpha\beta}(\boldsymbol{U}^x_{(\gamma\delta)})n_\beta\varphi_\alpha\,\mathrm{d}s. \tag{111}$$

On combining Eqs. (111) and (105), and using (106) one finds

$$A^{\alpha\beta\iota\varrho}\int_{\Omega}\epsilon_{\iota\varrho}(\boldsymbol{z}_{(\gamma\delta)}+\boldsymbol{U}^{x}_{(\gamma\delta)})\epsilon_{\alpha\beta}(\boldsymbol{\varphi})\,\mathrm{d}x=-\epsilon_{\gamma\delta}(\boldsymbol{\varphi})(\boldsymbol{0}). \tag{112}$$

Let us put $\boldsymbol{\varphi}=\boldsymbol{v}$ in (112) and $\tilde{\boldsymbol{v}}=\boldsymbol{z}_{(\gamma\delta)}+\boldsymbol{U}^{x}_{(\gamma\delta)}$ in (20). Hence

$$\int_{\partial\Omega}\boldsymbol{p}\cdot(\boldsymbol{z}_{(\gamma\delta)}+\boldsymbol{U}^{x}_{(\gamma\delta)})\,\mathrm{d}s=-\epsilon^{0}_{\gamma\delta}, \tag{113}$$

which is a consequence of Betti's theorem. Taking into account (110) one finds

$$\int_{\partial\Omega}\boldsymbol{p}\cdot\tilde{\boldsymbol{u}}\,\mathrm{d}s=-\epsilon^{0}_{\lambda\mu}M^{\lambda\mu\gamma\delta}\epsilon^{0}_{\gamma\delta}. \tag{114}$$

Substitution of (100) into (98) and making use of (114) gives the desired result (97). Since $\boldsymbol{M}$ is negative definite we note that $\mathscr{E}(\Omega_{\varepsilon})>\mathscr{E}(\Omega)$, irrespective of the shape of the hole ω_{ε}.

The energy change $\mathscr{E}(\Omega_{\varepsilon})-\mathscr{E}(\Omega)$ could alternatively be expressed in terms of the stress field $(\overset{\circ}{\sigma}{}^{\alpha\beta})$. Instead of (97) one can write

$$\mathscr{E}(\Omega_{\varepsilon})=\mathscr{E}(\Omega)+\frac{1}{2}\varepsilon^{2}\overset{\circ}{\sigma}{}^{\kappa\delta}H_{\kappa\delta\gamma\mu}\overset{\circ}{\sigma}{}^{\gamma\mu}+\mathrm{o}(\varepsilon^{2}), \tag{115}$$

where $\overset{\circ}{\sigma}{}^{\kappa\delta}=A^{\kappa\delta\gamma\mu}\epsilon^{0}_{\gamma\mu}$ and the tensor $\boldsymbol{H}$ is linked with the tensor $\boldsymbol{M}$ by

$$H_{\kappa\delta\gamma\mu}=-C_{\kappa\delta\alpha\beta}M^{\alpha\beta\lambda\nu}C_{\lambda\nu\gamma\mu} \tag{116}$$

or $\boldsymbol{H}=-\boldsymbol{CMC}$. Here $\boldsymbol{C}=\boldsymbol{A}^{-1}$ or

$$C_{\alpha\beta\lambda\mu}A^{\lambda\mu\gamma\delta}=\mathbb{I}^{\gamma\delta}_{\alpha\beta}. \tag{117}$$

Let us introduce new vector fields in $\mathbb{R}^{2}\setminus\overline{\varpi}$:

$$\boldsymbol{\Phi}_{(\sigma\gamma)}(y)=C_{\sigma\gamma\kappa\delta}\boldsymbol{\psi}^{(\kappa\delta)}(y), \tag{118}$$

where $\boldsymbol{\psi}^{(\kappa\delta)}$ are solutions of the problems $\widetilde{P}^{(\lambda\mu)}_{\omega}$ (cf. Section 2.2). The fields (118) are solutions to the following exterior problems

$$\hat{P}^{\omega}_{(\sigma\gamma)}\left|\begin{array}{ll}\text{find } \boldsymbol{\Phi}_{(\sigma\gamma)} \text{ defined in } \mathbb{R}^{2}\setminus\overline{\varpi} \text{ such that} & \\ A^{\alpha\beta\lambda\mu}\frac{\partial}{\partial y_{\beta}}\epsilon^{y}_{\lambda\mu}(\boldsymbol{\Phi}_{(\sigma\gamma)})=0 \quad \text{in } \mathbb{R}^{2}\setminus\overline{\varpi}, & (119)\\ A^{\alpha\beta\lambda\mu}\epsilon^{y}_{\lambda\mu}(\boldsymbol{\Phi}_{(\sigma\gamma)})\nu_{\beta}=0 \quad \text{on } \partial\omega, & (120)\\ \sigma^{\alpha\beta}_{y}(\boldsymbol{\Phi}_{(\sigma\gamma)})\to\mathbb{I}^{\alpha\beta}_{\sigma\delta} \quad \text{if } \|y\|\to\infty. & (121)\end{array}\right.$$

One easily notes that the formulation of the problems above is a direct consequence of the formulation $\widetilde{P}^{(\lambda\mu)}_{\omega}$, of the definition (118) and of the properties (117) and (118).

Substitution of (64) into (116) gives

$$H_{\kappa\delta\gamma\mu}=\int_{\partial\omega}\Phi_{(\gamma\mu)\kappa}\nu_{\delta}\,\mathrm{d}s. \tag{122}$$

By using symmetry properties (63), that refer also to the tensor $\boldsymbol{H}$, one can rearrange the above expression as follows:

$$H_{\kappa\delta\gamma\mu}=\frac{1}{2}\int_{\partial\omega}\left[\Phi_{(\gamma\mu)\kappa}\nu_{\delta}+\Phi_{(\gamma\mu)\delta}\nu_{\kappa}\right]\mathrm{d}s. \tag{123}$$

The following representation proves useful

$$\boldsymbol{H} = |\omega|\widetilde{\boldsymbol{H}}, \tag{124}$$

where dimensions of $\widetilde{\boldsymbol{H}}$ are the same as dimensions of tensor $\boldsymbol{C}$ of elastic flexibilities.

Solutions to the problems $\widehat{P}^{\omega}_{(\sigma\gamma)}$, for various shapes of ω, are reported in the known monographs on stress concentration around holes (see e.g. Savin, 1968). Upon finding the fields $\boldsymbol{\Phi}_{(\gamma\mu)}$ one can easy compute $(H_{\kappa\delta\gamma\mu})$ by (122).

Lastly, note that the energy change can be expressed in the form

$$\mathscr{E}(\Omega_\varepsilon) - \mathscr{E}(\Omega) = \frac{1}{2}\varepsilon^2|\Omega|\overset{\circ}{\sigma}{}^{\alpha\beta}\overset{\circ}{e}_{\alpha\beta}, \tag{125}$$

where

$$\overset{\circ}{e}_{\alpha\beta} = \frac{1}{2|\Omega|}\int_{\partial\omega}(u^0_\alpha \nu_\beta + u^0_\beta \nu_\alpha)\,\mathrm{d}s \tag{126}$$

with $\boldsymbol{u}^0 = \overset{\circ}{\sigma}{}^{\alpha\beta}\boldsymbol{\Phi}_{(\alpha\beta)}$.

3. The Eshelby method

The remote strain field given in a form of an mth order polynomial in $x = (x_1, x_2)$ induces in an ellipsoidal inclusion a strain field being also an mth order polynomial. This result of Kunin and Sosnina (1971) generalizes the result of Eshelby (1957) concerning the case of $m = 0$. This last result, along with R. Hill's concept of smearing-out non-homogeneities was a basement for the energy methods of assessing overall properties of composite materials.

Following Christensen (1979), Nemat-Nasser and Hori (1993) and Jasiuk et al. (1994, Eq. 35) we recall the Eshelby formula for the change of energy due to the appearance of a hole in a body Ω, subjected to stresses $\overset{\circ}{\sigma}{}^{\alpha\beta}$

$$\mathscr{E}(\Omega \setminus \omega) = \mathscr{E}(\Omega) + \frac{1}{2}\overset{\circ}{\sigma}{}^{\alpha\beta}\int_{\partial\omega} u^0_\alpha \nu_\beta\,\mathrm{d}s, \tag{127}$$

where $u^0_\alpha = (u^0_1, u^0_2)$ is the displacement field induced by $(\overset{\circ}{\sigma}{}^{\alpha\beta})$. One can write

$$\mathscr{E}(\Omega \setminus \omega) = \mathscr{E}(\Omega) + \frac{1}{2}\overset{\circ}{\sigma}{}^{\alpha\beta}\overset{\circ}{e}_{\alpha\beta}|\Omega|, \tag{128}$$

$$\overset{\circ}{e}_{\alpha\beta} = \frac{1}{2|\Omega|}\int_{\partial\omega}(u^0_\alpha \nu_\beta + u^0_\beta \nu_\alpha)\,\mathrm{d}s. \tag{129}$$

Let us emphasize a similarity between the formulae (128), (129) and (103), (125). The formula (128) is effective if the relation $\overset{\circ}{e}_{\alpha\beta}(\boldsymbol{\sigma}^0)$ is known. Due to linearity a tensor $\mathscr{H}$ exists such that

$$\overset{\circ}{e}_{\alpha\beta} = \mathscr{H}_{\alpha\beta\lambda\mu}\overset{\circ}{\sigma}{}^{\lambda\mu} \tag{130}$$

(see Nemat-Nasser and Hori, 1993). In the paper by Kachanov et al. (1994) the tensor $\mathscr{H}$ is called a *cavity compliance tensor*. Putting $\varepsilon = 1$ in (115) one finds

$$\mathscr{H} = \frac{1}{|\Omega|}\boldsymbol{H} \tag{131}$$

or

$$\mathscr{H} = p\widetilde{\boldsymbol{H}}, \quad p = \frac{|\omega|}{|\Omega|}, \tag{132}$$

cf. (124); p represents the area fraction of the hole and will be called porosity. Substitution of (130)–(132) into (128) gives

$$\mathscr{E}(\Omega \setminus \omega) = \mathscr{E}(\Omega) + \frac{1}{2} p|\Omega| \overset{\circ}{\sigma}{}^{\lambda\mu} \widetilde{H}_{\lambda\mu\alpha\beta} \overset{\circ}{\sigma}{}^{\alpha\beta}. \tag{133}$$

The tensors $\mathscr{H}$ and the Eshelby tensor $\boldsymbol{S}$, as defined in Mura (1982, Eq. (11.15)) and Nemat-Nasser and Hori (1993, Section 7.3.3) are interrelated by

$$\mathscr{H} = p(\mathbb{I} - \boldsymbol{S})^{-1} \boldsymbol{A}^{-1}. \tag{134}$$

Particular forms of this tensor for holes of different shapes are reported in Tsukrov and Kachanov (2000), Kachanov et al. (1994), Kachanov (1999), Sevostianov and Kachanov (1999), Shafiro and Kachanov (1999). The formula (134) holds both for the 3D and 2D cases.

4. Energy change due to the appearance of a circular hole. The topological derivative method for the plane elasticity problem

4.1. Topological derivative of shape functionals

The notion of a topological derivative has been introduced in Sokołowski and Żochowski (1999a,b) in order to formulate necessary conditions of optimality for optimum shape design problems. If the shape functional for the elasticity problem represents the compliance of the body, then the relevant topological derivative determines an infinitesimal change of energy, brought about by the appearance of a circular hole or a spherical cavity. Thus the theory of the topological derivative is linked with the problem of evaluating the change of energy due to the appearance of holes or cavities.

Assume that $J(\Omega)$ is a shape functional, while the shape design problem has the form

$$J(\Omega^*) = \inf_{\Omega} J(\Omega). \tag{135}$$

The optimal domain Ω^* satisfies the following set of necessary optimality conditions.

On the boundary $\partial\Omega^*$ of Ω^*

$$\mathrm{d}J(\Omega^*, \boldsymbol{V}) \geqslant 0 \tag{136}$$

for each admissible vector fields $\boldsymbol{V}$; the Eulerian semiderivative $\mathrm{d}J$ is explained in Sokołowski and Zolesio (1992).

In the interior of the domain Ω^*

$$\mathfrak{T}(x) \geqslant 0 \quad \text{in } \Omega^*. \tag{137}$$

Here $\mathfrak{T}(x)$ represents the topological derivative defined by

$$\mathfrak{T}(x) = \lim_{\varepsilon \searrow 0} \frac{J(\Omega \setminus \overline{\omega}_\varepsilon(x)) - J(\Omega)}{|\overline{\omega_\varepsilon(x)}|}, \tag{138}$$

where $\omega_\varepsilon(x)$ is here a circle of centre in $x \in \Omega$ and of radius εa. Thus $|\overline{\omega_\varepsilon(x)}| = \pi\varepsilon^2 a^2$. We refer the reader to Sokołowski and Żochowski (2001a,b) for the proof in the scalar case.

The same ideas can be applied to 3D shape optimization problems (cf. Sokołowski and Żochowski, 2001a,b).

4.2. Topological derivative of the energy functional

Assume that ω_ε is a circle defined above. Let $\boldsymbol{u}^\varepsilon$ be the unknown displacement field of the elasticity problem of Section 2.1. The potential energy of the body is given by the following shape functional

$$J(\Omega \setminus \overline{\omega}_\varepsilon) = \frac{1}{2}\int_{\Omega\setminus\overline{\omega}_\varepsilon} A^{\alpha\beta\lambda\mu}\epsilon_{\lambda\mu}(\boldsymbol{u}^\varepsilon)\epsilon_{\alpha\beta}(\boldsymbol{u}^\varepsilon)\,\mathrm{d}x - \int_{\partial\Omega}\boldsymbol{p}\cdot\boldsymbol{u}^\varepsilon\,\mathrm{d}s \tag{139}$$

the function $j(\varepsilon) = J(\Omega \setminus \overline{\omega}_\varepsilon)$ admits the following expansion (see Sokołowski and Żochowski, 1999a,b):

$$j(\varepsilon) = j(0) + \frac{1}{2}\varepsilon^2 j''(0^+) + \mathrm{o}(\varepsilon^2), \tag{140}$$

where $j(0) = J(\Omega)$ refers to the problem posed on the virgin domain

$$j(0) = -\frac{1}{2}\int_{\partial\Omega}\boldsymbol{p}\cdot\boldsymbol{v}\,\mathrm{d}s \tag{141}$$

with $\boldsymbol{v}$ being a solution to the problem (21)–(23). In the case of the body being isotropic the tensor $\boldsymbol{A}$ is given by (60) and the quantity $j''(0^+)$ equals

$$j''(0^+) = -\frac{\pi a^2}{\widetilde{E}}\left[(\mathring{\sigma}_{\mathrm{I}} + \mathring{\sigma}_{\mathrm{II}})^2 + 2(\mathring{\sigma}_{\mathrm{I}} - \mathring{\sigma}_{\mathrm{II}})^2\right], \tag{142}$$

where

$$\widetilde{E} = \frac{4k\mu}{k+\mu} \tag{143}$$

represents the effective Young modulus for the 2D setting. The quantities $\mathring{\sigma}_{\mathrm{I}}$, $\mathring{\sigma}_{\mathrm{II}}$ are principal stresses for the stress field $(\mathring{\sigma}^{\alpha\beta})$ (see (115)).

The expansion (140), along with (142), can be expressed in the form (115) with the tensor $\boldsymbol{H}$ assuming the following isotropic form

$$\boldsymbol{H} = \frac{2\pi a^2}{\widetilde{E}}(\boldsymbol{\Lambda}_1 + 2\boldsymbol{\Lambda}_2), \tag{144}$$

with $\boldsymbol{\Lambda}_\alpha$ defined by (7). The same formula for the energy change has been recently reported in Garreau et al. (2001), where a similar perturbation method has been used.

4.3. On perturbation of other shape functionals

Contrary to the compound asymptotics method and the Eshelby method, the topological derivative method makes it possible to examine perturbation of functionals other than the energy functional. The perturbation is understood as the appearance of a small circular hole within the domain considered. In particular, the topological derivatives of the following shape functionals were found in Sokołowski and Żochowski (1999a,b).

$$J_1(\Omega_\varepsilon) = \int_{\Omega_\varepsilon}(u^\varepsilon_\alpha B^{\alpha\beta}u^\varepsilon_\beta)^q\,\mathrm{d}x, \tag{145}$$

$$J_2(\Omega_\varepsilon) = \int_{\Omega_\varepsilon}\left[\sigma^{\alpha\beta}(\boldsymbol{u}^\varepsilon)\mathscr{R}_{\alpha\beta\lambda\mu}\sigma^{\lambda\mu}(\boldsymbol{u}^\varepsilon)\right]^q\,\mathrm{d}x, \tag{146}$$

$q = 1$ or $q = 2$; the matrices $\boldsymbol{B} \in \mathbb{M}^2_s$, $\mathscr{R} \in \mathbb{M}^4_s$ are assumed to be positive definite.

The topological derivatives of these functionals involve not only the field $\boldsymbol{v}$, but also an other field being a solution to the appropriate adjoint problem. In the case of the energy functional both the original and adjoint problems coincide. For the details the reader is referred to Sokołowski and Żochowski (1999a,b). An equivalent but different approach has been proposed by Garreau et al. (2001).

In the paper by Nazarov and Sokołowski (2003) the compound asymptotics method and the matched asymptotics method are applied to the so-called elliptic problems with the polynomial property. An approximation of the solution to such systems is defined with a prescribed precision which is controlled by the degree of certain polynomials. Such approximation is defined in the domain without any cavity; in the paper the space dimension $n \geqslant 3$ is fixed in order to simplify the presentation. Using the approximate solution in the domain with the cavity of the size $\varepsilon > 0$ it is possible to expand the shape functionals and identify the first term of the expansion which is exactly the topological derivative in the sense considered here provided that the Neumann type homogeneous boundary conditions are prescribed on the cavity. Such an approach is general and may be used to identify the form of the expansion for an arbitrary elliptic boundary value problem. The estimates in Hölder weighed spaces are also obtained for the remainder terms in the expansion.

5. Sensitivity of energy due to homothetic changes of shapes of holes and cavities

5.1. Sensitivity analysis of shape functionals: Case of holes and cavities of arbitrary shape

We come back to the setting of the problem considered in Section 2.1, where ε indexes the family of domains Ω_ε and the boundary value problems for the unknown $\boldsymbol{u}^\varepsilon$ field. We generalize the notion of the topological derivative of shape functionals by admitting the non-circular holes and non-spherical cavities. We modify the approach similar to that of Sokołowski and Żochowski (1999a,b) and assume that the formula (138) represents such a definition with ω being arbitrary. If ω is defined as in Section 2.1, then the definition (138) refers to the point $x = \mathbf{0}$. Possible shifting to other points from Ω is straightforward. Thus all holes or cavities $\omega_\varepsilon(x)$ are formed around the point $x \in \Omega$ and all of them are homothetic to each other. A similar approach for the Neumann problem has been proposed in Lewiński and Sokołowski (2000).

5.2. Topology derivative of the energy functional

The aim of this section is to find the topological derivative of the energy functional J for the 2D case ($n = 2$) and its counterpart for $n = 3$:

$$J(\Omega \setminus \overline{\omega}_\varepsilon) = \frac{1}{2} \int_{\Omega \setminus \overline{\omega}_\varepsilon} A^{ijkl} \epsilon_{ij}(\boldsymbol{u}^\varepsilon) \epsilon_{kl}(\boldsymbol{u}^\varepsilon) \, \mathrm{d}x - \int_{\partial\Omega} \boldsymbol{p} \cdot \boldsymbol{u}^\varepsilon \, \mathrm{d}S, \tag{147}$$

where Ω, ω_ε and ω are here 3D domains and $\boldsymbol{u}^\varepsilon$ solves the relevant 3D elasticity problem. As before $\mathbf{0} \in \omega_\varepsilon$ and ω_ε is defined by (24) for $x \in \mathbb{R}^n$, $n = 2, 3$.

It is clear that

$$|\omega_\varepsilon| = \varepsilon^n |\omega|, \qquad |\partial\omega_\varepsilon| = \varepsilon^{n-1} |\partial\omega|, \tag{148}$$

where $|\omega|$ and $|\partial\omega|$ represent the volume (area) and the area (length) of open sets ω and $\partial\omega$, for $n = 3$ or 2, respectively.

Let us consider the definition (138) in the case of $x = \mathbf{0}$; it means that a cavity or a hole appears at the point $x = \mathbf{0} \in \Omega$. Similarly to Section 4.2 we introduce the function $j(\varepsilon) = J(\Omega \setminus \overline{\omega}_\varepsilon)$, where ε should not exceed a value $\varepsilon_{\max}$ to fulfil the condition: $\overline{\omega}_\varepsilon \subset \Omega$.

By using (148) one can rearrange the definition (138) as follows:

$$\mathfrak{T}(\mathbf{0}) = \frac{1}{|\omega|} \lim_{\varepsilon \searrow 0^+} \frac{J(\Omega_\varepsilon) - J(\Omega)}{\varepsilon^n}. \tag{149}$$

For $\varepsilon = 0$ we denote $\Omega_{\varepsilon=0} = \Omega$.

Now we compute

$$\mathfrak{T}(\mathbf{0}) = \frac{1}{|\omega|} \lim_{\varepsilon \searrow 0^+} \frac{\mathrm{d}j(\varepsilon)}{\mathrm{d}(\varepsilon^n)} = \frac{1}{|\omega|} \frac{1}{n} \lim_{\varepsilon \searrow 0^+} \frac{1}{\varepsilon^{n-1}} \frac{\mathrm{d}j(\varepsilon)}{\mathrm{d}\varepsilon}. \tag{150}$$

Therefore, in order to compute $\mathfrak{T}(\mathbf{0})$ it is sufficient to determine $j'(\varepsilon)$ and find the limit: $\lim_{\varepsilon \searrow 0^+}(j'(\varepsilon)/\varepsilon^{n-1})$.

Evaluation of $j'(\varepsilon)$: To this end we use the velocity method of shape optimization (cf. Sokołowski and Zolesio, 1992). We recall that the shape derivative of the shape functional is defined in the following manner.

Let there be given a vector field $\boldsymbol{V}$ and the associated flow mapping: $T_\tau : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$T_\tau(\boldsymbol{V})(X) = x(\tau, X), \tag{151}$$

where

$$\begin{aligned} &\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = \boldsymbol{V}(\tau, x(\tau)), \\ &x(0) = X \end{aligned} \tag{152}$$

and the image of Ω_ε is denoted by $T_\tau(\Omega_\varepsilon)$. Then the Eulerian semiderivative of the shape functional $J(\Omega_\varepsilon)$ in the direction of the field $\boldsymbol{V}$ is defined by

$$\mathrm{d}J(\Omega_\varepsilon; \boldsymbol{V}) = \lim_{\tau \searrow 0} \frac{1}{\tau} [J(T_\tau(\Omega_\varepsilon)) - J(\Omega_\varepsilon)]. \tag{153}$$

To find $j'(\varepsilon)$ one should construct the mapping T_τ in such a way that $T_\tau(\Omega_\varepsilon) = \Omega_{\varepsilon+\tau}$. This means, in particular, that

$$\omega_{\varepsilon+\tau} = T_\tau(\omega_\varepsilon), \qquad \omega_\varepsilon = T_\tau^{-1}(\omega_{\varepsilon+\tau}) \tag{154}$$

for $|\tau| \leqslant (1/2)\varepsilon$. It is sufficient to choose the speed vector field in the form

$$\boldsymbol{V}(\tau, x) = \frac{x}{\varepsilon + \tau} \tag{155}$$

for $|\tau| \leqslant (1/2)\varepsilon$ and x lying in an open neighborhood of $\partial\omega_\varepsilon$. The field $\boldsymbol{V}$ is extended to $\mathbb{R}^n$ in such a way that it is a smooth vector function of both the arguments. Moreover, we assume that $\mathbf{0} \notin \operatorname{supp} \boldsymbol{V}$ and $\boldsymbol{V}$ vanishes along $\partial\Omega$.

Now we proceed to find $j'(\varepsilon)$. Let us first define the field $(\boldsymbol{u}^\varepsilon)'$ as follows:

$$(\boldsymbol{u}^\varepsilon)' = \lim_{\tau \searrow 0} \frac{1}{\tau} (\boldsymbol{u}^{\varepsilon+\tau} - \boldsymbol{u}^\varepsilon).$$

Here $\boldsymbol{u}^{\varepsilon+\tau}$ represents the solution to the problem (17) with Ω_ε replaced by $\Omega_{\varepsilon+\tau}$ when $n = 2$ and the case of $n = 3$ is treated in the same way. The shape derivative $(\boldsymbol{u}^\varepsilon)'$ solves the problem below (see Sokołowski and Zolesio, 1992): find $(\boldsymbol{u}^\varepsilon)'$ defined in Ω_ε such that

$$\int_{\Omega_\varepsilon} \boldsymbol{\sigma}((\boldsymbol{u}^\varepsilon)') : \boldsymbol{\epsilon}(\tilde{\boldsymbol{v}})\, \mathrm{d}x = \int_{\partial\omega_\varepsilon} \left[\Pi_\tau \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon) : \Pi_\tau(\nabla \tilde{\boldsymbol{v}})\right] \widehat{\boldsymbol{V}} \cdot \boldsymbol{\nu}\, \mathrm{d}S(x)$$

for sufficiently regular $\tilde{\boldsymbol{v}}$. Here $\Pi_\tau \boldsymbol{a}$ is a projection of $\boldsymbol{a} \in \mathbb{M}^2_s$ on the plane tangent to $\partial\omega_\varepsilon$. The quantity $\Pi_\tau(\nabla \boldsymbol{v})$ is in Sokołowski and Zolesio (1992) denoted by $\nabla_\tau \boldsymbol{v}$. Moreover, $\widehat{\boldsymbol{V}} = \boldsymbol{V}(0, x)$.

Let us recall the definition of $j(\varepsilon)$

$$j(\varepsilon) = -\frac{1}{2}\int_{\Omega_\varepsilon} \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon) : \boldsymbol{\epsilon}(\boldsymbol{u}^\varepsilon)\,\mathrm{d}x.$$

The first derivative of $j(\varepsilon)$ for $\varepsilon > 0$:

$$j'(\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\tau} J(T_\tau(\Omega_\varepsilon))\bigg|_{\tau=0}$$

is expressed by the following formula:

$$j'(\varepsilon) = -\int_{\Omega_\varepsilon} \boldsymbol{\sigma}((\boldsymbol{u}^\varepsilon)') : \boldsymbol{\epsilon}(\boldsymbol{u}^\varepsilon)\,\mathrm{d}x + \frac{1}{2}\int_{\partial\omega_\varepsilon} \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon) : \boldsymbol{\epsilon}(\boldsymbol{u}^\varepsilon)\widehat{\boldsymbol{V}} \cdot \boldsymbol{v}\,\mathrm{d}S(x).$$

The sign (+) in the last term follows from $\boldsymbol{v}$ being directed inward Ω_ε. By taking $\tilde{\boldsymbol{v}} = \boldsymbol{u}^\varepsilon$ in the variational equation for $(\boldsymbol{u}^\varepsilon)'$ and noting that

$$\Pi_\tau \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon(x)) : \Pi_\tau(\nabla \boldsymbol{u}^\varepsilon(x)) = \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon(x)) : \boldsymbol{\epsilon}(\boldsymbol{u}^\varepsilon(x)) \quad \text{for } x \in S$$

we simplify the expression for $j'(\varepsilon)$ to the form

$$j'(\varepsilon) = -\frac{1}{2}\int_{\partial\omega_\varepsilon} \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon) : \boldsymbol{\epsilon}(\boldsymbol{u}^\varepsilon)\widehat{\boldsymbol{V}} \cdot \boldsymbol{v}\,\mathrm{d}S(x). \tag{156}$$

Note that $\widehat{\boldsymbol{V}} = x/\varepsilon$ (see (155)). Moreover, $\mathrm{d}S(x) = \varepsilon^{n-1}\,\mathrm{d}S(y)$, $y = x/\varepsilon \in \partial\omega$.

Changing the variables in (156) one finds

$$j'(\varepsilon) = -\frac{1}{2}\varepsilon^{n-1}\int_{\partial\omega} \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon(\varepsilon y)) : \boldsymbol{\epsilon}(\boldsymbol{u}^\varepsilon(\varepsilon y))\boldsymbol{y} \cdot \boldsymbol{v}(y)\,\mathrm{d}S(y). \tag{157}$$

Evaluation of the limit for $\varepsilon \searrow 0^+$. Using the above expression and the result (150) one finds

$$\mathfrak{T}(\boldsymbol{0}) = -\frac{1}{2n|\omega|}\lim_{\varepsilon \searrow 0^+}\int_{\partial\omega} \boldsymbol{\sigma}(\boldsymbol{u}^\varepsilon(\varepsilon y)) : \boldsymbol{\epsilon}(\boldsymbol{u}^\varepsilon(\varepsilon y))\boldsymbol{y} \cdot \boldsymbol{v}(y)\,\mathrm{d}S(y). \tag{158}$$

This results in the following expansion of the shape functional $J(\Omega_\varepsilon)$ in the direction $\boldsymbol{V}$ such that $\boldsymbol{V} \cdot \boldsymbol{v} = \boldsymbol{y} \cdot \boldsymbol{v}$,

$$J(\Omega_\varepsilon) = J(\Omega) + \varepsilon^n|\omega|\mathfrak{T}(\boldsymbol{0}) + \mathrm{o}(\varepsilon^n). \tag{159}$$

In the 2D case ($n = 2$) we can specify the expression (158) by using the asymptotic result (56), to obtain

$$\mathfrak{T}(\boldsymbol{0}) = \frac{1}{2|\omega|}\epsilon^0_{\gamma\delta}G^{\gamma\delta\iota\varrho}\epsilon^0_{\iota\varrho}, \tag{160}$$

where the tensor $\boldsymbol{G}$ is defined by

$$G^{\gamma\delta\iota\varrho} = -\frac{1}{2}A^{\alpha\beta\lambda\mu}\int_{\partial\omega} \epsilon^y_{\alpha\beta}(\boldsymbol{\psi}^{(\iota\varrho)})\epsilon^y_{\lambda\mu}(\boldsymbol{\psi}^{(\gamma\delta)})\boldsymbol{y} \cdot \boldsymbol{v}\,\mathrm{d}s. \tag{161}$$

Upon substitution of (48) one can express the components of $\boldsymbol{G}$ as follows:

$$G^{\iota\varrho\gamma\delta} = -A^{\iota\varrho\gamma\delta}|\omega| - A^{\alpha\beta\lambda\mu}\int_{\partial\omega} \gamma^{(\iota\varrho)(\gamma\delta)}_{\alpha\beta\lambda\mu}(\boldsymbol{\chi})\boldsymbol{y} \cdot \boldsymbol{v}\,\mathrm{d}s \tag{162}$$

with

$$\gamma^{(\iota\varrho)(\gamma\delta)}_{\alpha\beta\lambda\mu}(\boldsymbol{\chi}) = \frac{1}{2}\left[\mathbb{I}^{\iota\varrho}_{\alpha\beta}\epsilon^{y}_{\lambda\mu}(\boldsymbol{\chi}^{(\gamma\delta)}) + \mathbb{I}^{\gamma\delta}_{\lambda\mu}\epsilon^{y}_{\alpha\beta}(\boldsymbol{\chi}^{(\iota\varrho)}) + \epsilon^{y}_{\alpha\beta}(\boldsymbol{\chi}^{(\iota\varrho)})\epsilon^{y}_{\lambda\mu}(\boldsymbol{\chi}^{(\gamma\delta)})\right]. \tag{163}$$

The functions $\boldsymbol{\chi}^{(\gamma\delta)}$ are solutions to the problems $P^{(\gamma\delta)}_{\omega}$ (Section 2.2). It can be shown that for sufficiently regular shapes of ω the directional derivative $\mathfrak{T}(\mathbf{0})$ becomes the first term of asymptotic expansion and it gives rise to the expansion of the elastic energy

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) - \frac{1}{2}\varepsilon^2\epsilon^0_{\alpha\beta}G^{\alpha\beta\lambda\mu}\epsilon^0_{\lambda\mu} + \mathrm{o}(\varepsilon^2). \tag{164}$$

Let $\boldsymbol{W} = \epsilon^0_{\lambda\mu}\boldsymbol{\psi}^{(\lambda\mu)}$. Then the formula (164) assumes the form

$$\mathscr{E}(\Omega_\varepsilon) - \mathscr{E}(\Omega) = \frac{1}{2}\varepsilon^2 A^{\alpha\beta\lambda\mu}\int_{\partial\omega}\epsilon^{y}_{\alpha\beta}(\boldsymbol{W})\epsilon^{y}_{\lambda\mu}(\boldsymbol{W})\boldsymbol{y}\cdot\boldsymbol{v}\,\mathrm{d}s + \mathrm{o}(\varepsilon^2). \tag{165}$$

By virtue of the positive definiteness property (12) one can estimate

$$\mathscr{E}(\Omega_\varepsilon) - \mathscr{E}(\Omega) \geqslant \frac{c}{2}\varepsilon^2\int_{\partial\omega}\|\boldsymbol{\epsilon}^{y}(\boldsymbol{W})\|^2\boldsymbol{y}\cdot\boldsymbol{v}\,\mathrm{d}s. \tag{166}$$

Note that $\boldsymbol{\psi}^{(\lambda\mu)} \notin \mathscr{R}$, hence $\boldsymbol{W} \notin \mathscr{R}$. Let us recall that the domains ω for which $\boldsymbol{y}\cdot\boldsymbol{v} > 0$, possibly with exception of corner points of $\partial\omega$, are called *star-shaped*. Thus the star-shaped property implies that $\mathscr{E}(\Omega_\varepsilon) > \mathscr{E}(\Omega)$. In other cases the estimate (166) is not helpful. Let us recall that the formula (97) provided us with a stronger result of the difference $\mathscr{E}(\Omega_\varepsilon) - \mathscr{E}(\Omega)$ being positive irrespective of the shape of $\partial\omega$. Now we arrive at two seemingly different estimates of change of energy ((97) and (164)). Fortunately, there is no ambiguity here. Only recently Nazarov and Sokołowski (2003) proved that $\boldsymbol{G} = \boldsymbol{M}$ for arbitrary shapes of $\partial\omega$. Since the proof is being published in the paper mentioned above it is sufficient to give only an outline of the proof. Let us recall the operator

$$\sigma^{\alpha\beta}_{y}(\boldsymbol{u}) = A^{\alpha\beta\lambda\mu}\epsilon^{y}_{\lambda\mu}(\boldsymbol{u})$$

and let Γ represent any contour encircling the contour $\partial\omega$. The proof is based upon the contour integral

$$\frac{1}{2}\int_{\Gamma}[v_\alpha\sigma^{\alpha\beta}_{y}(\boldsymbol{u})n_\beta - u_\alpha\sigma^{\alpha\beta}_{y}(\boldsymbol{v})n_\beta]\,\mathrm{d}s$$

being path independent. We choose $\boldsymbol{u} = \boldsymbol{\psi}^{(\gamma\delta)}$ and $\boldsymbol{v} = y_\alpha(\partial/\partial y_\alpha)(\boldsymbol{\psi}^{(\iota\rho)})$.

First this integral is evaluated along $\partial\omega$ by multiple using of the properties (49) and (50). The result assumes the form (161). Then, this integral is evaluated along a circle of radius R and eventually computed for R tending to infinity, by using asymptotic behavior of $\boldsymbol{\psi}^{(\alpha\beta)}$ at infinity. The result occurs to be equal to $M^{\gamma\delta\iota\rho}$. This proves the equality $\boldsymbol{M} = \boldsymbol{G}$.

5.3. A link to the bubble method by Eschenauer et al.

In this section we show that the expression (160) defines the characteristic function of the *bubble method* (see Eschenauer et al., 1994). The plane problem is considered ($n = 2$).

By using (118) one can rearrange the formula (164) as follows:

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) + \frac{1}{2}\varepsilon^2\mathring{\sigma}^{\alpha\beta}\widehat{H}_{\alpha\beta\lambda\mu}\mathring{\sigma}^{\lambda\mu} + \mathrm{o}(\varepsilon^2) \tag{167}$$

with $\widehat{\boldsymbol{H}} = -\boldsymbol{CGC}$ or

$$\widehat{H}_{\iota\varrho\gamma\delta} = \frac{1}{2} C_{\alpha\beta\lambda\mu} \int_{\partial\omega} \sigma_y^{\alpha\beta}(\boldsymbol{\Phi}_{(\iota\varrho)}) \sigma_y^{\lambda\mu}(\boldsymbol{\Phi}_{(\gamma\delta)}) \boldsymbol{y} \cdot \boldsymbol{v}\, \mathrm{d}s. \tag{168}$$

The boundary condition (120) makes the above formula simpler

$$\widehat{H}_{\iota\varrho\gamma\delta} = \frac{1}{2} \int_{\partial\omega} C\tau\tau\tau\tau(s) \sigma_y^{\tau\tau}(\boldsymbol{\Phi}_{(\iota\varrho)}) \sigma_y^{\tau\tau}(\boldsymbol{\Phi}_{(\gamma\delta)}) \boldsymbol{y} \cdot \boldsymbol{v}\, \mathrm{d}s \tag{169}$$

with $C\tau\tau\tau\tau(s)$ and $\sigma^{\tau\tau}$ being the components of $\boldsymbol{C}$ and $\boldsymbol{\sigma}$ referred to the basis $(\boldsymbol{v}, \boldsymbol{\tau})$ along $\partial\omega$. One notes that $\widehat{\boldsymbol{H}}$ depends only on the hoop stresses associated with the fields $\boldsymbol{\Phi}_{(\alpha\beta)}$. The equality $\boldsymbol{M} = \boldsymbol{G}$ implies $\boldsymbol{H} = \widehat{\boldsymbol{H}}$ for all shapes of ω.

In the case of isotropy the hoop stresses $\sigma^{\tau\tau}(\boldsymbol{\Phi}_{(\alpha\beta)})$ are independent of both the moduli k and μ, which follows from the well-known Michell's theorem.

Let us note that the quadratic form $\mathring{\sigma}^{\alpha\beta} \widehat{H}_{\alpha\beta\lambda\mu} \mathring{\sigma}^{\lambda\mu}$ differs in a factor from the characteristic function of the bubble method (see Schumacher, 1995, Eq. (4.11)).

6. Elliptical hole in an isotropic body: Plane problem

6.1. Exact solutions to the exterior problems $\widehat{P}^{\omega}_{(\sigma\gamma)}$

We consider a plane, infinite isotropic and homogeneous body of moduli k and μ (see (60)) occupying an exterior of the ellipse ω whose contour is given by

$$\left(\frac{y_1}{a}\right)^2 + \left(\frac{y_2}{b}\right)^2 = 1. \tag{170}$$

Here a and b are half-lengths of the major and minor axes of the ellipse. Let

$$l = \frac{1}{2}(a+b), \quad c = \frac{a-b}{a+b}, \quad a > b, \quad c \in [0, 1).$$

Assume that $\zeta = \varrho \mathrm{e}^{\mathrm{i}\vartheta}$. Note that the function

$$y_1 + \mathrm{i} y_2 = \wp(\zeta) = l\left(\zeta + \frac{c}{\zeta}\right) \tag{171}$$

transforms the exterior of the unit circle in the complex plane onto the domain $\mathbb{R}^2 \setminus \overline{\omega}$.

The aim of the present section is to recall the formulae for the components of the fields $\boldsymbol{\Phi}_{(\sigma\gamma)}$ being solutions to the problems $\widehat{P}^{\omega}_{(\sigma\gamma)}$ of Section 2.4.

Let us start with recalling the solution to the problem of a sheet with the elliptical hole (170) subjected to remote stresses of intensity q acting in the direction α with respect to the axis y_1. The complex potentials read, [2] (see Muskhelishvili, 1975, Section 82a)

$$\varphi(\zeta) = \frac{1}{4} q l \left[\zeta + \frac{1}{\zeta}(2\mathrm{e}^{2\mathrm{i}\alpha} - c)\right], \tag{172}$$

$$\psi(\zeta) = -\frac{1}{2} q l \left[\mathrm{e}^{-2\mathrm{i}\alpha}\zeta + \frac{1}{c\zeta}\mathrm{e}^{2\mathrm{i}\alpha} - \frac{(1+c^2)}{c} \frac{\zeta}{\zeta^2 - c}(\mathrm{e}^{2\mathrm{i}\alpha} - c)\right]. \tag{173}$$

[2] Potential $\psi(\zeta)$ has nothing to do with the fields $\psi_{\gamma}^{(\alpha\beta)}$.

The displacements (u_1, u_2) measured along (y_1, y_2) are given by

$$2\mu(u_1 + \mathrm{i}u_2) = \varkappa\varphi(\zeta) - \wp(\zeta)\frac{\overline{\varphi'(\zeta)}}{\overline{\wp'(\zeta)}} - \overline{\psi(\zeta)} \tag{174}$$

with $\varkappa = 1 + 2\mu/k$. The hoop stresses $\sigma^{\tau\tau}$ are expressed by

$$\sigma^{\tau\tau}\big|_{\partial\omega} = 4\mathrm{Re}(F(\zeta))\big|_{\zeta=\mathrm{e}^{\mathrm{i}\vartheta}}, \qquad F(\zeta) = \frac{\varphi'(\zeta)}{\wp'(\zeta)}. \tag{175}$$

The fields $\Phi_{(\sigma\gamma)}$ can now be expressed by the solution (174) as follows:

$$\begin{aligned}
&\Phi_{(11)} = (u_1, u_2)\big|_{q=1,\ \alpha=0}, \\
&\Phi_{(22)} = (u_1, u_2)\big|_{q=1,\ \alpha=\pi/2}, \\
&\Phi_{(12)} = \Phi_{(21)} = (u_1, u_2)\big|_{q=1/2,\ \alpha=\pi/4} + (u_1, u_2)\big|_{q=-1/2,\ \alpha=-\pi/4}.
\end{aligned} \tag{176}$$

The components of the fields $\Phi_{(\sigma\gamma)}$, measured along $\partial\omega$, are given by the formulae (A.1) (see Appendix A). The associated hoop stresses read as follows:

$$\begin{aligned}
&\sigma_{\tau\tau}(\Phi_{(11)})\big|_{\partial\omega} = [(1 + 2c - c^2) - 2\cos 2\vartheta]f(\vartheta), \\
&\sigma_{\tau\tau}(\Phi_{(12)})\big|_{\partial\omega} = -2\sin 2\vartheta f(\vartheta), \\
&\sigma_{\tau\tau}(\Phi_{(22)})\big|_{\partial\omega} = [(1 - 2c - c^2) + 2\cos 2\vartheta]f(\vartheta)
\end{aligned} \tag{177}$$

with

$$f(\vartheta) = (1 + c^2 - 2c\cos 2\vartheta)^{-1}. \tag{178}$$

6.2. Components of tensor $\boldsymbol{M}$

By (60) we have

$$A^{1111} = k + \mu, \quad A^{1122} = k - \mu, \quad A^{2222} = k + \mu, \quad A^{1212} = \mu.$$

Thus the relations inverse to (118) are expressed as follows:

$$\begin{aligned}
&\boldsymbol{\psi}^{(11)} = (k + \mu)\Phi_{(11)} + (k - \mu)\Phi_{(22)}, \\
&\boldsymbol{\psi}^{(12)} = 2\mu\Phi_{(12)}, \\
&\boldsymbol{\psi}^{(22)} = (k - \mu)\Phi_{(11)} + (k + \mu)\Phi_{(22)}.
\end{aligned} \tag{179}$$

The expanded form of (64) read

$$\begin{aligned}
&M^{1111} = -(k + \mu)\int_{\partial\omega} \psi_1^{(11)} v_1 \,\mathrm{d}s - (k - \mu)\int_{\partial\omega} \psi_2^{(11)} v_2 \,\mathrm{d}s, \\
&M^{1122} = -(k + \mu)\int_{\partial\omega} \psi_1^{(22)} v_1 \,\mathrm{d}s - (k - \mu)\int_{\partial\omega} \psi_2^{(22)} v_2 \,\mathrm{d}s, \\
&M^{2222} = -(k + \mu)\int_{\partial\omega} \psi_2^{(22)} v_2 \,\mathrm{d}s - (k - \mu)\int_{\partial\omega} \psi_1^{(22)} v_1 \,\mathrm{d}s, \\
&M^{1212} = -\mu\int_{\partial\omega} \left(\psi_1^{(12)} v_2 + \psi_2^{(12)} v_1\right)\mathrm{d}s
\end{aligned} \tag{180}$$

and the remaining components are determined by symmetry properties (63).

Let us prove the expressions

$$\nu_1\,\mathrm{d}s = l(1-c)\cos\vartheta\,\mathrm{d}\vartheta, \qquad \nu_2\,\mathrm{d}s = l(1+c)\sin\vartheta\,\mathrm{d}\vartheta. \tag{181}$$

We note that $\nu_1 = \cos\alpha$ and $\nu_2 = \sin\alpha$. Moreover (see Muskhelishvili, 1975, Section 49)

$$\mathrm{e}^{-\mathrm{i}\alpha} = \frac{1}{|\wp'(\zeta)|}\bar{\zeta}\,\overline{\wp'(\zeta)} \tag{182}$$

and $\mathrm{d}s = |\wp'(\zeta)|\,\mathrm{d}\vartheta$ along the contour $\varrho = 1$. Thus

$$\begin{aligned} \nu_1\,\mathrm{d}s &= \mathrm{Re}[\mathrm{e}^{\mathrm{i}\vartheta}\wp'(\zeta)]\big|_{\zeta=\mathrm{e}^{\mathrm{i}\vartheta}}\,\mathrm{d}\vartheta, \\ \nu_2\,\mathrm{d}s &= \mathrm{Re}[-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\vartheta}\wp'(\zeta)]\big|_{\zeta=\mathrm{e}^{\mathrm{i}\vartheta}}\,\mathrm{d}\vartheta. \end{aligned} \tag{183}$$

Taking into account the form of $\wp$ (see (171)) we arrive at (181).

Substitution of (A.1) into (179) and further into (180), and taking into account (181) leads to some integrals over ϑ. By using the results of integration gathered in (A.3) one finds the following closed formulae for the non-zero components ($M^{\alpha\beta\lambda\mu}$)

$$\begin{aligned} M^{1111} &= -\frac{\pi l^2(k+\mu)}{k\mu}[(1+c^2)k^2 - 4ck\mu + 2\mu^2], \\ M^{1122} &= -\frac{\pi l^2(k+\mu)}{k\mu}[(1+c^2)k^2 - 2\mu^2], \\ M^{2222} &= -\frac{\pi l^2(k+\mu)}{k\mu}[(1+c^2)k^2 + 4ck\mu + 2\mu^2], \\ M^{1212} &= -2\pi l^2\frac{(k+\mu)\mu}{k}. \end{aligned} \tag{184}$$

Let us compute

$$M^{1111}M^{2222} - (M^{1122})^2 = 8\pi^2 l^4(1-c^2)(k+\mu)^2$$

since $c \in [0,1)$ we conclude that the quadratic form $q_{\alpha\beta}M^{\alpha\beta\lambda\mu}q_{\lambda\mu}$, $\boldsymbol{q} \in \mathbb{M}^2_s$, is negative definite. The results (184) agree with those derived in Movchan and Movchan (1995, Section 5.1.3) directly from the very definition (57).

Remark. In the case of an elliptical hole the complex potentials assume closed form expressions (172) and (173). In general, for a hole of arbitrary shape these potentials are expanded in series. Recently Argatov (1998, Section 2) showed how to express the components of tensor $\boldsymbol{M}$ in terms of the coefficients of these series. Alternatively, the components of $\boldsymbol{M}$ can be expressed in terms of the coefficients of the conformal mapping, see Argatov (1998, Section 3).

6.3. Components of tensor $\widehat{\boldsymbol{H}}$

To find the components of the tensor $\widehat{\boldsymbol{H}}$ the formula (169) will be used. Let us begin with proving that

$$\boldsymbol{y}\cdot\boldsymbol{\nu}\,\mathrm{d}s = l^2(1-c^2)\,\mathrm{d}\vartheta. \tag{185}$$

Indeed, by the formula

$$\boldsymbol{y}\cdot\boldsymbol{\nu} = \mathrm{Re}[(y_1 + \mathrm{i}y_2)\,\mathrm{e}^{-\mathrm{i}\alpha}], \tag{186}$$

where $y_1 + \mathrm{i}y_2 = \wp(\zeta)$, with using (182) and by taking into account that $\mathrm{d}s = |\wp'(\zeta)|\,\mathrm{d}\vartheta$ for $\varrho = 1$, one finds

$$\boldsymbol{y}\cdot\boldsymbol{v}\,\mathrm{d}s = \mathrm{Re}\Big[\overline{\zeta}\,\overline{\wp'(\zeta)}\wp(\zeta)\Big]\Big|_{\zeta=\mathrm{e}^{\mathrm{i}\vartheta}}\mathrm{d}\vartheta. \tag{187}$$

For $\wp(\zeta)$ of the form (171) we arrive at (185).

In the case of elliptical domain ω the formula (169) assumes the form

$$\widehat{H}_{\sigma\gamma\omega\varrho} = \frac{(1-c^2)l^2}{2\widetilde{E}}\int_0^{2\pi}\sigma^{\tau\tau}(\boldsymbol{\Phi}_{(\sigma\gamma)})\sigma^{\tau\tau}(\boldsymbol{\Phi}_{(\omega\varrho)})\,\mathrm{d}\vartheta \tag{188}$$

with $\widetilde{E}$ defined by (143). By substitution of (177) into (188) and using the integral formulae (A.4) (see Appendix A) one finds

$$\begin{aligned}
&\widehat{H}_{1111} = \frac{\pi l^2}{\widetilde{E}}(1-c)(3-c),\\
&\widehat{H}_{2222} = \frac{\pi l^2}{\widetilde{E}}(1+c)(3+c),\\
&\widehat{H}_{1122} = -\frac{\pi l^2}{\widetilde{E}}(1-c^2),\\
&\widehat{H}_{2211} = \widehat{H}_{1122},\\
&\widehat{H}_{1212} = \frac{2\pi l^2}{\widetilde{E}},\\
&\widehat{H}_{1221} = \widehat{H}_{2112} = \widehat{H}_{2121} = \widehat{H}_{1212}
\end{aligned} \tag{189}$$

and other components vanish.

6.4. Components of tensors $\boldsymbol{G}$ and $\boldsymbol{H}$

The formula $\widehat{\boldsymbol{H}} = -\boldsymbol{CGC}$ (see Section 5.2) implies $\boldsymbol{G} = -\boldsymbol{A}\widehat{\boldsymbol{H}}\boldsymbol{A}$. Due to isotropy of tensor $\boldsymbol{A}$ this formula reduces to

$$G^{\lambda\mu\zeta\eta} = -(k-\mu)^2\widehat{H}_{\sigma\omega}^{\sigma\ \ \omega}\delta^{\lambda\mu}\delta^{\zeta\eta} - 4\mu^2\widehat{H}^{\mu\lambda\zeta\eta} - 2\mu(k-\mu)(\delta^{\zeta\eta}\widehat{H}^{\mu\lambda}{}_{\omega}{}^{\omega} + \delta^{\lambda\mu}\widehat{H}^{\sigma}{}_{\sigma}{}^{\zeta\eta}). \tag{190}$$

Substitution of (189) results in

$$G^{\alpha\beta\lambda\mu} = M^{\alpha\beta\lambda\mu}, \tag{191}$$

where $(M^{\alpha\beta\lambda\mu})$ are given by (184). Consequently

$$H_{\alpha\beta\lambda\mu} = \widehat{H}_{\alpha\beta\lambda\mu}, \tag{192}$$

where $\boldsymbol{H} = -\boldsymbol{CMC}$ (see (116)). Thus we have confirmed that in the case of an elliptical opening we have

$$\boldsymbol{G} = \boldsymbol{M} \quad \text{and} \quad \boldsymbol{H} = \widehat{\boldsymbol{H}}. \tag{193}$$

Note yet that G^{1212} and H_{1212} are independent of the ratio a/b.

6.5. Case of a circular hole: Unification of all approaches

Let us specify the previous results for the case of a circular opening of radius a. Here $c = 0$. Tensors $\boldsymbol{G} = \boldsymbol{M}$ and $\boldsymbol{H} = \widehat{\boldsymbol{H}}$ assume the isotropic form

$$\boldsymbol{G} = \pi a^2(2\xi\boldsymbol{\Lambda}_1 + 2\eta\boldsymbol{\Lambda}_2), \tag{194}$$

$$\boldsymbol{H} = \frac{2\pi a^2}{\widetilde{E}}(\boldsymbol{\Lambda}_1 + 2\boldsymbol{\Lambda}_2) \tag{195}$$

with

$$\xi = -\frac{(k+\mu)k}{\mu}, \quad \eta = -2\frac{(k+\mu)\mu}{k} \tag{196}$$

and $\widetilde{E}$ is given by (143). The formulae (144) and (195) coincide. We conclude that in the case of a circular opening all examined methods of assessing the change of energy result in the same formula (115) with tensor $\boldsymbol{H}$ given by (195). The energy of a body weakened by a circular opening of radius εa reads

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) + \frac{\pi(\varepsilon a)^2}{2\widetilde{E}}[(\mathrm{tr}\,\boldsymbol{\sigma}^0)^2 + 4\|\boldsymbol{s}^0\|^2] + \mathrm{o}(\varepsilon^2), \tag{197}$$

where $\boldsymbol{s}^0 = \boldsymbol{\sigma}^0 - (1/2)\mathrm{tr}\,\boldsymbol{\sigma}^0\mathbb{1}$ represents the deviator of $\boldsymbol{\sigma}^0$. Alternatively

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) + \frac{\pi(\varepsilon a)^2}{2\widetilde{E}}[(\boldsymbol{\sigma}_{\mathrm{I}}^0 + \boldsymbol{\sigma}_{\mathrm{II}}^0)^2 + 2(\boldsymbol{\sigma}_{\mathrm{I}}^0 - \boldsymbol{\sigma}_{\mathrm{II}}^0)^2] + \mathrm{o}(\varepsilon^2), \tag{198}$$

where $\boldsymbol{\sigma}_{\mathrm{I}}^0$, $\boldsymbol{\sigma}_{\mathrm{II}}^0$ are principal stresses.

7. The cavity problem in three dimensions

7.1. The Eshelby method

The formulae of Section 3 have their counterparts in the 3D problem. Then the opening ω is called cavity, its surface is denoted by $\partial\omega$ and its volume (by $|\omega|$). In the case of ω being an ellipsoid the tensor $\mathscr{H}$ is given by (134), where $\boldsymbol{S}$ is Eshelby's tensor (see Mura, 1982).

Assume that the body is homogeneous and isotropic, with the elastic moduli tensor represented by

$$\boldsymbol{A} = 3K\boldsymbol{\Lambda}_1 + 2\mu\boldsymbol{\Lambda}_2. \tag{199}$$

Here $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$ are projection operators (see (4)). The bulk (K) and shear (μ) moduli are linked with Young's modulus E and Poisson's ratio v by

$$3K = \frac{E}{1-2v}, \qquad 2\mu = \frac{E}{1+v}. \tag{200}$$

Thus, tensor $\boldsymbol{C} = \boldsymbol{A}^{-1}$ is represented by

$$\boldsymbol{C} = \frac{1-2v}{E}\boldsymbol{\Lambda}_1 + \frac{1+v}{E}\boldsymbol{\Lambda}_2 \tag{201}$$

see (6). Let us confine our attention to the case of a spherical cavity of radius a. The Eshelby tensor has the following isotropic representation

$$\boldsymbol{S} = \frac{1}{3(1+v)}\left[(1+v)\boldsymbol{\Lambda}_1 + \frac{2}{5}(4-5v)\boldsymbol{\Lambda}_2\right]. \tag{202}$$

Let us compute

$$\mathbb{I} - \boldsymbol{S} = \frac{2}{3}\frac{1-2v}{1-v}\boldsymbol{\Lambda}_1 + \frac{7-5v}{15(1-v)}\boldsymbol{\Lambda}_2 \tag{203}$$

and by using the properties (5) one finds

$$\boldsymbol{A}(\mathbb{I}-\boldsymbol{S}) = 2K\frac{1-2\nu}{1-\nu}\boldsymbol{\Lambda}_1 + 2\mu\frac{7-5\nu}{15(1-\nu)}\boldsymbol{\Lambda}_2 \tag{204}$$

which enables one to find tensor $\mathscr{H}$ (see (134))

$$\mathscr{H} = p\left[\frac{1-\nu}{2(1-2\nu)K}\boldsymbol{\Lambda}_1 + \frac{15(1-\nu)}{2\mu(7-5\nu)}\boldsymbol{\Lambda}_2\right], \tag{205}$$

where $p = (4/3)\pi a^3/|\Omega|$. By (131) one computes the tensor $\boldsymbol{H}$

$$\boldsymbol{H} = \frac{2(1-\nu)}{E}\pi a^3\left[\boldsymbol{\Lambda}_1 + \frac{10(1+\nu)}{7-5\nu}\boldsymbol{\Lambda}_2\right] \tag{206}$$

which determines the change of energy in the form

$$\frac{1}{2}\mathring{\sigma}^{ij}H_{ijkl}\mathring{\sigma}^{kl} = \frac{1-\nu}{7-5\nu}\frac{\pi a^3}{E}[-(1+5\nu)(\operatorname{tr}\mathring{\boldsymbol{\sigma}})^2 + 10(1+\nu)\|\mathring{\boldsymbol{\sigma}}\|^2] \tag{207}$$

(cf. Kachanov et al., 1994, Eq. (5.16)).

The tensors $\boldsymbol{S}$ and $\boldsymbol{H}$ assume peculiar forms in the case of $\nu = 1/5$. Then $\boldsymbol{S} = (1/2)\mathbb{I}$ and

$$\boldsymbol{H}|_{\nu=1/5} = \frac{8}{5}\frac{\pi a^3}{E}(\boldsymbol{\Lambda}_1 + 2\boldsymbol{\Lambda}_2). \tag{208}$$

The form above is similar to that of the 2D case (cf. (195)).

7.2. The compound asymptotics method

The compound asymptotics method recalled in Section 2 in the 2D context, generalizes to the 3D setting (cf. Mazja et al., 1991). Although the Somigliana solutions assume different forms, the representation (57) holds with only the error term corrected; it reads now $0(\|y\|^{-3})$, $y = (y_1, y_2, y_3)$.

The exterior boundary value problems $\widetilde{P}_\omega^{(\lambda\mu)}$ have the following 3D counterparts

$$\widetilde{P}_\omega^{(mn)} \left| \begin{array}{ll} \text{find } \boldsymbol{\psi}^{(mn)} \text{ given in } \mathbb{R}^3\backslash\overline{\omega} \text{ such that} & \\ A^{ijkl}\frac{\partial}{\partial y_j}\epsilon^y_{kl}(\boldsymbol{\psi}^{(mn)}) = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\omega}, & (209) \\ A^{ijkl}\epsilon^y_{kl}(\boldsymbol{\psi}^{(mn)})\nu_j = 0 \quad \text{on } \partial\omega, & (210) \\ \boldsymbol{\psi}^{(mn)} \to \boldsymbol{E}^{(mn)} \quad \text{if } \|y\| \to \infty. & (211) \end{array}\right.$$

Here $\|y\|^2 = (y_1)^2 + (y_2)^2 + (y_3)^2$ and

$$\boldsymbol{E}^{(mn)}(y) = \frac{1}{2}(y_m\boldsymbol{e}_n + y_n\boldsymbol{e}_m). \tag{212}$$

Tensor $\boldsymbol{M}$ is defined by the formula

$$M^{mnpq} = -A^{ijmn}\int_{\partial\omega}\psi^{(pq)}_{(i}\nu_{j)}\,\mathrm{d}S \tag{213}$$

similar to (64). The components of tensor $\boldsymbol{H}$ read (cf. (122))

$$H_{mnpq} = \int_{\partial\omega}\boldsymbol{\Phi}_{(pq)(m}\nu_{n)}\,\mathrm{d}S, \tag{214}$$

where $\mathbf{\Phi}_{(pq)}(y) = C_{pqmn}\boldsymbol{\psi}^{(mn)}(y)$ are solutions to the problems $\widehat{P}^{\omega}_{(pq)}$ analogous to the problems $\widehat{P}^{\omega}_{(\sigma\gamma)}$. The tensors $\boldsymbol{M}$ and $\boldsymbol{H}$ are interrelated by $\boldsymbol{H} = -\boldsymbol{CMC}$.

By analogy with the representation (48) we write

$$\boldsymbol{\psi}^{(mn)} = \boldsymbol{\chi}^{(mn)} + \boldsymbol{E}^{(mn)}, \tag{215}$$

where the functions $\boldsymbol{\chi}^{(mn)}$ are solutions to the problems

$$P^{(mn)}_{\omega} \left| \begin{array}{ll} \text{find } \boldsymbol{\chi}^{(mn)} \text{ defined in } \mathbb{R}^3\backslash\overline{\varpi} \text{ such that} & \\ A^{ijkl}\frac{\partial}{\partial y_j}\epsilon^y_{kl}(\boldsymbol{\chi}^{(mn)}) = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\varpi}, & (216) \\ A^{ijkl}\epsilon^y_{kl}(\boldsymbol{\chi}^{(mn)})v_j = -A^{ijmn}v_j \quad \text{on } \partial\omega, & (217) \\ \boldsymbol{\chi}^{(mn)} \to \mathbf{0} \quad \text{if } \|y\| \to \infty. & (218) \end{array} \right.$$

The definition (213) can be put in the form

$$M^{mnpq} = -(A^{mnpq}|\omega| + \mathscr{M}^{mnpq}) \tag{219}$$

with

$$\mathscr{M}^{mnpq} = A^{mnij}\int_{\partial\omega}\chi^{(pq)}_{(i}v_{j)}\,\mathrm{d}S. \tag{220}$$

The appearance of a cavity ω_ε brings about an increase of elastic energy according to the formulae

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) - \frac{1}{2}\varepsilon^3\epsilon^0_{ij}M^{ijkl}\epsilon^0_{kl} + \mathrm{o}(\varepsilon^3), \tag{221}$$

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) + \frac{1}{2}\varepsilon^3\mathring{\sigma}^{ij}H_{ijkl}\mathring{\sigma}^{kl} + \mathrm{o}(\varepsilon^3) \tag{222}$$

analogous to those known from the 2D case (cf. (97) and (115)).

In the case of a spherical cavity of radius a the tensor $\boldsymbol{H}$, which appeared in (222), assumes the form (206). This identification requires an additional proof which will be omitted here (cf. Sokołowski and Żochowski, 1999a,b).

7.3. Sensitivity of energy functional due to homothetic changes of cavities

Let us specify the result (158) to the 3D case ($n = 3$). The topological derivative of J equals

$$\mathfrak{T}(\mathbf{0}) = \frac{1}{2|\omega|}\epsilon^0_{ij}G^{ijkl}\epsilon^0_{kl}, \tag{223}$$

where

$$G^{ijkl} = -\frac{1}{3}A^{mnpq}\int_{\partial\omega}\epsilon^y_{mn}(\boldsymbol{\psi}^{(ij)})\epsilon^y_{pq}(\boldsymbol{\psi}^{(kl)})\boldsymbol{y}\cdot\boldsymbol{v}\,\mathrm{d}S. \tag{224}$$

The relation $J(\Omega_\varepsilon) = -\mathscr{E}(\Omega_\varepsilon)$ implies the following expansion for the elastic energy

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) - \frac{1}{2}\varepsilon^3\epsilon^0_{ij}G^{ijkl}\epsilon^0_{kl} + \mathrm{o}(\varepsilon^3) \tag{225}$$

or

$$\mathscr{E}(\Omega_\varepsilon) = \mathscr{E}(\Omega) + \frac{1}{2}\varepsilon^3 \mathring{\sigma}^{ij}\widehat{H}_{ijkl}\mathring{\sigma}^{kl} + \mathrm{o}(\varepsilon^3), \tag{226}$$

where $\widehat{\boldsymbol{H}} = -\boldsymbol{CGC}$ and

$$\widehat{H}_{ijmn} = \frac{1}{3}C_{pqrs}\int_{\partial\omega} \sigma_y^{pq}(\boldsymbol{\Phi}_{(ij)})\sigma_y^{rs}(\boldsymbol{\Phi}_{(mn)})\boldsymbol{y}\cdot\boldsymbol{v}\,\mathrm{d}S. \tag{227}$$

Here $\boldsymbol{\Phi}_{(mn)} = C_{mnkl}\boldsymbol{\psi}^{(kl)}$ represent displacement fields caused by remote unit stress states, cf. problems $\widehat{P}^{\omega}_{(\sigma\gamma)}$ of Section 2.4, Eqs. (119)–(121). Tensor $\boldsymbol{G}$ given by (224) is equal to tensor $\boldsymbol{M}$ defined by (219) (see Nazarov and Sokołowski, 2003). The proof is similar to that for the 2D case, as outlined in Section 5.2; the contour integral becomes a surface integral. Consequently, the tensors $\boldsymbol{H}$ and $\widehat{\boldsymbol{H}}$ coincide.

In the case of isotropy and of ω having a spherical shape the tensor $\hat{\boldsymbol{H}} = \boldsymbol{H}$ assumes the form (206). The same result has been recently confirmed in Garreau et al. (2001).

8. Effective properties of solids with cavities

The problem of assessing effective moduli of porous media has been extensively developed in the literature. Despite these ample sources one can notice no attempts to interrelate the Eshelby-like results (see Nemat-Nasser and Hori, 1993), the asymptotic results of Mazja et al. (1991) and the homogenization results for porous media (see Jikov et al., 1994, Lewiński and Telega, 2000, Section 3.12). The aim of this section is to show such relationships.

8.1. The homogenization approach

Consider a homogeneous elastic body of moduli (A^{mnpq}) weakened by periodically distributed cavities ω_ε. The periodicity cell is a parallelepiped $Y_\varepsilon = \varepsilon Y$, $Y = [0, l_1] \times [0, l_2] \times [0, l_3]$ and $\varepsilon > 0$ is a small parameter. Each periodicity cell is weakened by the cavity ω_ε. Thus the material cells are $Y \setminus \overline{\omega}_\varepsilon$. The rescaled periodicity cell $Y \setminus \overline{\omega}$ is parameterized by the Cartesian coordinates y_1, y_2, y_3. Let $\boldsymbol{v}$ be a unit vector outward normal to $\partial\omega$ and let $\boldsymbol{n}$ be a unit vector outward normal to ∂Y.

According to the homogenization approach the effective properties of a porous body are determined by solutions to the following basic cell problems:

$P^{(ij)}_{\mathrm{loc}}$ | find $Y\setminus\overline{\omega}$—periodic vector fields $\boldsymbol{T}^{(ij)}$ such that

$$A^{mnpq}\frac{\partial}{\partial y_n}\epsilon^y_{pq}(\boldsymbol{T}^{(ij)}) = 0 \quad \text{in } Y\setminus\overline{\omega}, \tag{228}$$

$$A^{mnpq}\epsilon^y_{pq}(\boldsymbol{T}^{(ij)})v_n = -A^{mnij}v_n \quad \text{on } \partial\omega, \tag{229}$$

$$A^{mspq}\epsilon^y_{pq}(\boldsymbol{T}^{(ij)})n_s \text{ assume opposite values at opposite sides of } Y\setminus\overline{\omega}. \tag{230}$$

The effective or homogenized moduli are expressed by

$$A_H^{klmn} = \frac{1}{|Y|}\int_{Y\setminus\overline{\omega}}\left[A^{klmn} + A^{klpq}\epsilon^y_{pq}(\boldsymbol{T}^{(mn)})\right]\mathrm{d}y. \tag{231}$$

The conditions (230) make it possible to rearrange the above formula as follows:

$$A_H^{klmn} = \frac{|Y\setminus\overline{\omega}|}{|Y|}A^{klmn} - \frac{1}{|Y|}A^{klpq}\int_{\partial\omega}T_p^{(mn)}v_q\,\mathrm{d}S. \tag{232}$$

The negative sign above follows from the vector $\boldsymbol{v}$ being directed outward normal to $\partial\omega$ and hence inward the domain $Y \setminus \overline{\omega}$. For future references we rewrite (232) in the form

$$A_H^{klmn} = A^{klmn} - \frac{1}{|Y|}\left(|\omega|A^{klmn} + A^{klpq}\int_{\partial\omega} T_p^{(mn)} v_q \, \mathrm{d}S\right). \tag{233}$$

8.2. Dilute distribution of cavities: Homogenization approach

Let ω be located in the centre of Y. Assume, moreover, that dimensions of ω are much smaller than the dimensions l_1, l_2, l_3, of Y. In such a case the fields $\boldsymbol{T}^{(ij)}$ decay from the boundary $\partial\omega$ and almost vanish on ∂Y. Consequently, the fields $\boldsymbol{T}^{(ij)}$ cease to depend on $Y \setminus \overline{\omega}$; they depend rather on ω, exclusively. Thus one can assume that $\boldsymbol{T}^{(ij)}$ are sought in the domain $\mathbb{R}^3 \setminus \overline{\omega}$ and the periodicity conditions (230) are replaced by radiation conditions:

$$\boldsymbol{T}^{(ij)}(y) \to 0 \quad \text{if } \|y\| \to \infty. \tag{234}$$

Now we note that under the assumption of ω being small the fields $\boldsymbol{T}^{(ij)}$ can be identified with $\boldsymbol{\chi}^{(ij)}$, the solutions of $P_\omega^{(ij)}$ (see Section 7.2). Thus the formula (233) assumes the form

$$A_H^{klmn} = A^{klmn} - \frac{1}{|Y|}(|\omega|A^{klmn} + \mathscr{M}^{klmn}), \tag{235}$$

where $\mathscr{M}$ has been defined by (220). Using notation (219) one can write

$$\boldsymbol{A}_H = \boldsymbol{A} + \frac{1}{|Y|}\boldsymbol{M}. \tag{236}$$

Let us define

$$J_{ij}^{pq} = \frac{1}{2|Y|}\int_{\partial\omega}\left(\psi_i^{(pq)} v_j + \psi_j^{(pq)} v_i\right)\mathrm{d}S, \tag{237}$$

where $\boldsymbol{\psi}^{(pq)}$ are solutions of $(\widetilde{P}_\omega^{(pq)})$ (see Section 7.2). We note that (236) can be put in the form

$$\boldsymbol{A}_H = \boldsymbol{A} - \boldsymbol{A}\boldsymbol{J} \tag{238}$$

well known from the book by Nemat-Nasser and Hori (1993, Eq. (4.5.5a)). This links the homogenization approach with Eshelby-like approach in which the overall microstrain is prescribed.

Let us apply the formula (236) to determine effective elastic moduli of a body weakened by a dilute distribution of spherical cavities. Since $\boldsymbol{H} = -\boldsymbol{CMC}$ we determine tensor $\boldsymbol{M}$ by $\boldsymbol{M} = -\boldsymbol{AHA}$ and $\boldsymbol{H}$ is given by (206). Hence $\boldsymbol{M}$ can be written in the form

$$\boldsymbol{M} = -(3\alpha K\boldsymbol{\Lambda}_1 + 2\mu\beta\boldsymbol{\Lambda}_2)|\omega|, \tag{239}$$

where

$$\alpha = \frac{3}{2}\frac{1-v}{1-2v}, \qquad \beta = \frac{15(1-v)}{7-5v}. \tag{240}$$

Due to random distribution of cavities the tensor $\boldsymbol{A}_H$ is predicted in the isotropic form

$$\boldsymbol{A}_H = 3K_H\boldsymbol{\Lambda}_1 + 2\mu_H\boldsymbol{\Lambda}_2. \tag{241}$$

Thus the formulae (236), (239) and (241) imply

$$K_H = (1 - \alpha p)K, \qquad \mu_H = (1 - \beta p)\mu, \tag{242}$$

where $p = |\omega|/|Y|$ represents porosity density. The formulae above coincide with those derived in Nemat-Nasser and Hori (1993, Eqs. (5.2.5b) and (8.1.9a,b)) by the method of prescribing the microstrain. The same formulae were found by Eshelby (1957, p. 390) by linearizing the following formulae

$$K_* = (1 + \alpha p)^{-1}K, \qquad \mu_* = (1 + \beta p)^{-1}\mu \tag{243}$$

with respect to p (cf. also Hlavaček, 1986).

The formulae (242) are equivalent to the following expressions for the effective Young modulus E_H and Poisson ratio ν_H

$$\frac{E_H}{E} = \frac{\left[1 - \frac{15(1-\nu)}{7-5\nu}p\right]\left[1 - \frac{3}{2}\frac{1-\nu}{1-2\nu}p\right]}{1 - \frac{3(1-\nu)(5\nu^2 - 6\nu + 4)}{(7-5\nu)(1-2\nu)}p}, \tag{244}$$

$$\nu_H = \nu - \frac{15(1-\nu^2)(\nu - 1/5)}{2(7-5\nu)} \frac{p}{1 - \frac{3(1-\nu)(5\nu^2-6\nu+4)}{(1-2\nu)(7-5\nu)}p}. \tag{245}$$

In the case of $\nu = 1/5$ we have $\alpha = \beta = 2$ and

$$\frac{E_H}{E} = 1 - 2p, \quad \nu_H = 1/5. \tag{246}$$

Linearization of (244) and (245) with respect to p gives

$$\begin{aligned} \frac{E_H}{E} &= 1 - \frac{3}{2}\frac{(-5\nu^2 - 4\nu + 9)}{(7-5\nu)}p + 0(p^2), \\ \nu_H &= \nu - \frac{3}{2}\frac{(1-\nu^2)(5\nu - 1)}{7-5\nu}p + 0(p^2). \end{aligned} \tag{247}$$

8.3. Eshelby-like approach to the dilute distribution of cavities

Let us apply the formula (133) to the basic cell $Y \setminus \overline{\varpi}$. The elastic energy accumulated in the basic cell is represented by

$$\mathscr{E}(Y \setminus \overline{\varpi}) = \mathscr{E}(Y) + \frac{1}{2}p|Y|\mathring{\sigma}^{ij}\widetilde{H}_{ijkl}\mathring{\sigma}^{kl}, \tag{248}$$

where

$$\mathscr{E}(Y) = \frac{1}{2}|Y|\mathring{\sigma}^{ij}C_{ijkl}\mathring{\sigma}^{kl} \tag{249}$$

and $p = |\omega|/|Y|$. By (205) and (131) we have

$$\widetilde{\boldsymbol{H}} = \frac{\alpha}{3K}\boldsymbol{\Lambda}_1 + \frac{\beta}{2\mu}\boldsymbol{\Lambda}_2 \tag{250}$$

with α, β given by (240). Thus $\overset{\circ}{\sigma}^{ij}$ are treated here as remote stresses. The energy $\mathscr{E}(Y \setminus \overline{\varpi})$ is approximated by $(1/2)|Y|\overset{\circ}{\sigma}^{ij}C_{*ijkl}\overset{\circ}{\sigma}^{kl}$ with $\boldsymbol{C}_*$ treated as isotropic

$$\boldsymbol{C}_* = \frac{1}{3K_*}\boldsymbol{\Lambda}_1 + \frac{1}{2\mu_*}\boldsymbol{\Lambda}_2. \tag{251}$$

Since the equality (248) holds for all $(\overset{\circ}{\sigma}^{ij})$ we obtain $\boldsymbol{C}_* = \boldsymbol{C} + p\widetilde{\boldsymbol{H}}$, hence

$$\frac{1}{K_*} = \frac{1}{K}(1 + \alpha p), \qquad \frac{1}{\mu_*} = \frac{1}{\mu}(1 + \beta p) \tag{252}$$

which is equivalent to Eshelby's formulae (243). The derivation of (252) can be found in Nemat-Nasser and Hori (1993), Jasiuk et al. (1994) and Kachanov et al. (1994). The effective Young modulus and Poisson ratio are expressed by

$$\begin{aligned} E_* &= E\left[1 + \frac{3(1-v)(5v+9)}{2(7-5v)}p\right]^{-1}, \\ v_* &= \frac{1}{5} + (v - 1/5)\frac{1 + \dfrac{6(1-v)}{7-5v}p}{1 + \dfrac{3(1-v)(5v+9)}{2(7-5v)}p}. \end{aligned} \tag{253}$$

Note that the above formula for v_* coincides with that reported in Kachanov et al. (1994) upon correction of a misprint in the numerator.

In the case of $v = 1/5$ we have

$$\begin{aligned} &\frac{E_*}{E} = (1+2p)^{-1}, \quad v_* = 1/5, \\ &\frac{K_*}{E} = (1+2p)^{-1}, \quad \frac{\mu_*}{\mu} = (1+2p)^{-1}. \end{aligned} \tag{254}$$

Linearization of (253) with respect to p gives the previous results (247).

Let us compare the effective moduli found in Sections 8.2 and 8.3. We note that

$$K_* - K_H > 0, \quad \mu_* - \mu_H > 0, \quad E_* - E_H > 0 \tag{255}$$

and

$$\begin{aligned} &E_* - E_H = 0(p^2), \quad v_* - v_H = 0(p^2), \\ &K_* - K_H = 0(p^2), \quad \mu_* - \mu_H = 0(p^2). \end{aligned} \tag{256}$$

The moduli K_H and μ_H are linearization of K_* and μ_*, respectively. Moreover,

$$\begin{aligned} &v_* > v_H \quad \text{if } v \in (0, 1/5), \\ &v_* < v_H \quad \text{if } v > 1/5. \end{aligned} \tag{257}$$

Remark. Effective moduli for the 2D problem

Let us find k_*, μ_* as well as

$$\widetilde{E}_* = \frac{4k_*\mu_*}{k_* + \mu_*}, \quad \nu_* = \frac{k_* - \mu_*}{k_* + \mu_*} \tag{258}$$

(cf. (143)). We have here

$$\boldsymbol{C} = \frac{1}{2k}\boldsymbol{\Lambda}_1 + \frac{1}{2\mu}\boldsymbol{\Lambda}_2, \quad \boldsymbol{C}_* = \frac{1}{2k_*}\boldsymbol{\Lambda}_1 + \frac{1}{2\mu_*}\boldsymbol{\Lambda}_2, \quad \widetilde{H} = \frac{2}{\widetilde{E}}(\boldsymbol{\Lambda}_1 + 2\boldsymbol{\Lambda}_2). \tag{259}$$

Here $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Lambda}_2$ are defined by (7). The equality: $\boldsymbol{C}_* = \boldsymbol{C} + p\widetilde{\boldsymbol{H}}$ results in

$$\begin{aligned} &\frac{1}{k_*} = \frac{1}{k} + \frac{4p}{\widetilde{E}}, \quad \frac{1}{\mu_*} = \frac{1}{\mu} + \frac{8p}{\widetilde{E}}, \\ &\widetilde{E}_* = \frac{\widetilde{E}}{1+3p}, \quad \nu_* = \frac{\nu + p}{1+3p} \end{aligned} \tag{260}$$

(see Nemat-Nasser and Hori, 1993, Section 5.1; Jasiuk et al., 1994, Eq. (38)). Linearization of (256) gives Eqs. (40) and (41) in Jasiuk et al. (1994), where the negative sign at α should be replaced by (+).

8.4. On identification of cavities by boundary measurements

The approximation of shape functionals

$$\mathscr{J}(\Omega_\varepsilon) = \mathscr{J}(\Omega) + \mathscr{M}_n(\omega_\varepsilon)\mathscr{T}(x) + \mathrm{o}(\mathscr{M}_n(\omega_\varepsilon))$$

can be used in shape optimization as well as for the numerical solution of some inverse problems. In the above formula $\mathscr{M}_n(\omega_\varepsilon)$ measures the size of $\omega_\varepsilon \subset \mathbb{R}^n$, $n = 2, 3$, and $\mathscr{T}(x)$ denotes the topological derivative of the shape functional $\mathscr{J}(\Omega)$ evaluated at $x \in \Omega$,

$$\mathscr{T}(x) = \lim_{\varepsilon \to 0^+} \frac{\mathscr{J}(\Omega_\varepsilon) - \mathscr{J}(\Omega)}{\mathscr{M}_n(\omega_\varepsilon)}.$$

If the Neumann homogeneous boundary conditions are prescribed on the cavity, then $\mathscr{M}_n(\omega_\varepsilon) = (\mathrm{diam}\,\omega_\varepsilon)^n$ can be selected in the case of integral functionals defined in Ω. The case of boundary integrals, better suited for the inverse problems with boundary measurements is treated in Nazarov and Sokołowski (2003). The information given by the approximation of shape functional is used in Jackowska-Strumiłło et al. (1999) to perform the learning process of an artificial neural network. The results of computations for 2D examples in the case of a scalar elliptic equation show, that the method allows to determine an approximation of the global solution to the inverse problem with circular opening, sufficiently closed to the exact solution. It means that we can identify the localization and the size of the circular opening that is determined by three numbers. The proposed method can be extended to the problems with an opening of general shape and to the identification problems of small inclusions. However, the mathematical theory of the proposed approach still requires further research. In particular, the proof of global convergence of the method is an open problem.

9. Concluding remarks and open problems

Let us draw our attention to the following points

(i) the Somigliana solutions determine the tensor $\boldsymbol{M}$ of Polya and Szegö (see (57));
(ii) tensor $\boldsymbol{M}$ conditions the increment of energy caused by the appearance of a small hole (see (97)) or cavity (see (221));

(iii) tensor $\boldsymbol{H}$ is linked with tensor $\boldsymbol{M}$ by (116). Its components are determined by solutions to the classical stress-concentration problems (see (119)–(123)). The tensor $\boldsymbol{H}$ is linked with the Eshelby tensor $\boldsymbol{S}$ by (133). For a circular hole tensor $\boldsymbol{H}$ assumes an isotropic form (see (143)) and for a spherical cavity, its isotropic representation is given by (205);

(iv) the sensitivity approach leads to the tensor $\widehat{\boldsymbol{H}}$ given by (168) and (217) for the 2D and 3D cases, respectively. Its components coincide with the components of tensor $\boldsymbol{H}$;

(v) in the 2D setting a plane opening can be conformally mapped onto a unit circle. The tensor $\boldsymbol{H}$ is fixed by two first terms of this mapping. These two terms determine an effective elliptical hole. On the other hand, the formula for tensor $\widehat{\boldsymbol{H}}$ involves all terms of the conformal mapping. Nevertheless the equality $\widehat{\boldsymbol{H}} = \boldsymbol{H}$ shows that further terms of the conformal mapping do not contribute to $\widehat{\boldsymbol{H}}$;

(vi) the formula for $\widehat{\boldsymbol{H}}$ suggests that this tensor is positive definite for star-shaped domains ω only. The equality $\widehat{\boldsymbol{H}} = \boldsymbol{H}$ proves that in fact $\widehat{\boldsymbol{H}}$ is unconditionally positive definite;

(vii) the classical, or strain-controlled homogenization formulae lead, in a dilute approximation, to the Eshelby-like formulae reported in Nemat-Nasser and Hori (1993). Upon such approximation the homogenized tensor becomes explicitly dependent on the Polya–Szegö tensor $\boldsymbol{M}$. For the case of spherical cavities one obtains the effective bulk and shear moduli as decaying functions, linear with respect to the porosity density (see (242)). Both the decaying functions coincide for $\nu = 1/5$;

(viii) the stress-controlled Eshelby-like approach, as exploited in Nemat-Nasser and Hori (1993), Jasiuk et al. (1994) and Kachanov et al. (1994), leads, within the dilute approximation, to the decaying functions (252) for the effective bulk and shear moduli. Both these functions coincide for $\nu = 1/5$.

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Appendix A

The displacement fields (176) are expressed as follows. Let $F^{\alpha}_{\beta\gamma} = (2\mu/l)\Phi^{\alpha}_{(\beta\gamma)}\big|_{\partial\omega}$. We have

$$
\begin{aligned}
F^1_{11} &= a_1 \cos\vartheta + f(\vartheta)[a_2 \cos\vartheta + a_3 \cos\vartheta \cos 2\vartheta + a_4 \sin\vartheta \sin 2\vartheta],\\
F^2_{11} &= b_1 \sin\vartheta + f(\vartheta)[b_2 \sin\vartheta + b_3 \sin\vartheta \cos 2\vartheta + b_4 \cos\vartheta \sin 2\vartheta],\\
F^1_{22} &= c_1 \cos\vartheta + f(\vartheta)[c_2 \cos\vartheta + c_3 \cos\vartheta \cos 2\vartheta + c_4 \sin\vartheta \sin 2\vartheta],\\
F^2_{22} &= d_1 \sin\vartheta + f(\vartheta)[d_2 \sin\vartheta + d_3 \sin\vartheta \cos 2\vartheta + d_4 \cos\vartheta \sin 2\vartheta],\\
F^1_{12} &= \left(1 + \frac{\mu}{k} + \frac{1}{2c}\right)\sin\vartheta - \frac{1}{2c} f(\vartheta)[-c \sin 3\vartheta + (1 + c + c^2)\sin\vartheta],\\
F^2_{12} &= \left(1 + \frac{\mu}{k} - \frac{1}{2c}\right)\cos\vartheta + \frac{1}{2c} f(\vartheta)[-c \cos 3\vartheta + (1 - c + c^2)\cos\vartheta],
\end{aligned}
\tag{A.1}
$$

where $f(\vartheta)$ is given by (178) and the coefficients a_i, b_i, c_i, d_i read

$$
\begin{aligned}
&a_1 = \frac{\varkappa}{4}(3-c) + \frac{1}{2}\frac{1+c}{c}, \quad a_2 = -\frac{1+c}{4}(1+2c-c^2) - \frac{1}{2c}(1+c^2)(1-c)^2,\\
&a_3 = \frac{1}{2}(1+c), \quad a_4 = -\frac{1}{2}(1-c)^2,\\
&b_1 = \frac{\varkappa}{4}(c-1) + \frac{1}{2}\frac{1-c}{c}, \quad b_2 = -\frac{1-c}{4}(1+2c-c^2) - \frac{1}{2c}(1+c^2)(1-c^2),\\
&b_3 = \frac{1}{2}(1-c), \quad b_4 = \frac{1}{2}(1-c^2),\\
&c_1 = -\frac{\varkappa}{4}(c+1) - \frac{1}{2}\frac{1+c}{c}, \quad c_2 = -\frac{1}{4}(1+c)(1-2c-c^2) + \frac{1}{2c}(1+c^2)(1-c^2),\\
&c_3 = -\frac{1}{2}(1+c), \quad c_4 = \frac{1}{2}(1-c^2),\\
&d_1 = \frac{\varkappa}{4}(3+c) - \frac{1}{2}\frac{1-c}{c}, \quad d_2 = -\frac{1}{4}(1-c)(1-2c-c^2) + \frac{1}{2c}(1+c^2)(1+c)^2,\\
&d_3 = -\frac{1}{2}(1-c), \quad d_4 = -\frac{1}{2}(1+c)^2.
\end{aligned}
\tag{A.2}
$$

The definite integrals involved in Eq. (180) are expressed by

$$
\begin{aligned}
&\int_0^{2\pi} \cos^2\vartheta f(\vartheta)\,\mathrm{d}\vartheta = \frac{\pi}{1-c},\\
&\int_0^{2\pi} \sin^2\vartheta f(\vartheta)\,\mathrm{d}\vartheta = \frac{\pi}{1+c},\\
&\int_0^{2\pi} \cos^2\vartheta \cos 2\vartheta f(\vartheta)\,\mathrm{d}\vartheta = \frac{\pi}{2}\frac{1+c}{1-c},\\
&\int_0^{2\pi} \sin^2\vartheta \cos 2\vartheta f(\vartheta)\,\mathrm{d}\vartheta = -\frac{\pi}{2}\frac{1-c}{1+c},\\
&\int_0^{2\pi} \sin^2 2\vartheta f(\vartheta)\,\mathrm{d}\vartheta = \pi
\end{aligned}
\tag{A.3}
$$

if $c \in [0,1)$. The definite integrals necessary to determine the components (189) have the form

$$
\begin{aligned}
&\int_0^{2\pi} \sin^2 2\vartheta f^2(\vartheta)\,\mathrm{d}\vartheta = \frac{\pi}{1-c^2},\\
&\int_0^{2\pi} (1+2c-c^2-2\cos 2\vartheta)^2 f^2(\vartheta)\,\mathrm{d}\vartheta = \frac{2\pi(3-c)}{1+c},\\
&\int_0^{2\pi} (1-2c-c^2+2\cos 2\vartheta)(1+2c-c^2-2\cos 2\vartheta) f^2(\vartheta)\,\mathrm{d}\vartheta = -2\pi,\\
&\int_0^{2\pi} (1-2c-c^2+2\cos 2\vartheta)^2 f^2(\vartheta)\,\mathrm{d}\vartheta = \frac{2\pi(3+c)}{1-c}.
\end{aligned}
\tag{A.4}
$$

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